Middle East Technical University Fall 2000 Math 537 Algebraic Topology I

HOMEWORK I

HOMOTOPY AND CW-COMPLEXES

Problem 1. *Prove that the following conditions are equivalent*

(1) *X is contractible,*
(2) *any map $Y \to X$ is null-homotopic,*
(3) *any map $X \to Y$ is null-homotopic.*

Problem 2. *Consider space $X = S^2/p = q$ obtained by gluing together a pair of points, p and q, on a sphere S^2. Then an arc, $a \subset S^2$ connecting p and q becomes a loop $\bar{a} \subset X$. Attaching a 2-cell along $\bar{a}$ we obtain a 2-complex Y. Describe a simplest (minimal) homotopy model of Y (i.e., a CW complex homotopic to Y with the minimal number of cells).*

Observe the relation between Y and $S^1 \wedge S^1$.

Problem 3. *Given any map $f\colon X \to Y$ and a space Z,*

(1) *define the induced map $f_*\colon \pi(Z, X) \to \pi(Z, Y)$.*
(2) *Prove that $f_* = g_*$ if f and g are homotopic.*
(3) *Deduce from the latter that f_* is a bijection if f is a homotopy equivalence.*
(4) *What can be said about f_* if f is a retraction ?*

Problem 4. *Prove that $X * Y$ is homotopy equivalent to $S(X \wedge Y)$. Hint: construct a homeomorphism between $X * Y/X * \{y_0\} \cup \{x_0\} * Y$ and $S(X \wedge Y)/S(\{x_0\} \wedge \{y_0\})$ using that both $X * Y$ and $S(X \wedge Y)$ are quotients of $X \times Y \times I$.*

Problem 5. *Prove that $Z = X \cup Y$ is contractible provided CW-complexes X, Y as well as their common CW-subcomplex $X \cap Y$ are contractible.*

Problem 6. *By definition, the infinitely-dimensional sphere S^∞ is the union $\bigcup_{n>0} S^n$. Prove that S^∞ is contractible. Define the contraction either by formulae, or describing the trajectories of the points of S^∞ in the process of contraction.*

Problem 7. *Prove homotopy invariance of the constructions of a product $X \times Y$, join $X * Y$, wedge of spaces $X \vee Y$, smash-product $X \wedge Y$ and suspension SX. More precisely, prove that*

(1) *The constructions involving basepoints do not depend of its choice within a path connected component of the space. (So, it makes sense to omit the basepoint in the notation, at least for path connected spaces.)*
(2) *If X, Y are replaced by homotopy equivalent spaces, then the results of all the constructions will be the same up to homotopy equivalence.*

Problem 8. *Assume that CW complexes X, Y are n-connected. Prove that*

(1) *$X \times Y$ is n-connected,*
(2) *$X \vee Y$ is n-connected,*
(3) *the suspension $S(X)$ is $(n+1)$-connected,*
(4) *$X \wedge Y$ is $(2n)$-connected,*
(5) *$X * Y$ is $(2n+1)$-connected.*

Hint: you may use the results of the previous problem.

Problem 9. *Assume that* X, Y *are CW complexes,* $A \subset X$ *is a subcomplex,* $\dim A \leqslant n$ *and* Y *is* n*-connected. Prove that* $X \cup_f Y \cong (X/A) \vee Y$ *for any mapping* $f\colon A \to Y$.