Constant Coefficient Linear Homogeneous Reccurrence Relations.

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_t a_{n-t}$$

is called a constant coefficient, linear, homogeneous recurrence relation of order t, provided $c_t \neq 0$, where $c_1, \ldots, c_t$ are (constant) real numbers.

In this lecture, we will learn how to find the general solution of such a relation.

First we give examples of recurrence relations which are not of this form.

$a_n = 3a_{n-1} + n a_{n-2} + a_{n-3}$ is not constant coefficient

$a_n = 4a_{n-1}^2 + a_{n-2} + 4a_{n-3}a_{n-4}$ is not linear

$a_n = 2a_{n-1} + 3a_{n-3} - 5$ is not homogeneous

At the end of this section you will be able to solve any const. coeff. lin. hom. relation of any order. But, we start with relations of order 2. After completing the theory for order 2, we will generalize the results to the general case, order t.

> In this section, from now on, by a 'relation' or a 'recurrence relation' we will mean a const coeff. lin. hom. recurrence relation.

CONST. COEFF. LIN. REL. OF ORDER 2.

SET OF SOLUTIONS OF A RELATION OF ORDER 2.

Given a relation:

$$a_n = \alpha a_{n-1} + \beta a_{n-2}. \qquad (\star)$$

① - Let $\{u_n\}$ be a solution of $(\star)$, then for any $k \in \mathbb{R}$, $\{k u_n\}$ is also a solution.

$$k(u_n) = k(\alpha u_{n-1} + \beta u_{n-2})$$
$$\Rightarrow \quad (k u_n) = \alpha(k u_{n-1}) + \beta(k u_{n-2}) \checkmark$$

② - Let $\{u_n\}$ and $\{v_n\}$ be solutions of $(\star)$, then $\{w_n\} = \{u_n + v_n\}$ is also a solution:

$$w_n = u_n + v_n$$
$$= \underbrace{(\alpha u_{n-1} + \beta u_{n-2})}_{\{u_n\} \text{ solution}} + \underbrace{(\alpha v_{n-1} + \beta v_{n-2})}_{\{v_n\} \text{ solution}}$$
$$= \alpha(u_{n-1} + v_{n-1}) + \beta(u_{n-2} + v_{n-2})$$
$$= \alpha w_{n-1} + \beta w_{n-2}.$$

③ - The zero sequence $\{0_n\} = 0 \; 0 \; 0 \; 0 \; \ldots$ is a solution of $(\star)$.

④- Let $\{u_n\}: u_0 \; u_1 \; u_2 \; u_3 \; \cdots$ and
$\{v_n\}: v_0 \; v_1 \; v_2 \; v_3 \; \cdots$
be solutions of $(\bigstar)$.

From property ②, we know that sum of these solutions is also a solution. Further, combining with ①, we see that any linear combination, that is a sum $\{k u_n + \ell v_n\}$ $(k, \ell \in \mathbb{R})$, of these solutions is also a solution.

Moreover, any solution of $(\bigstar)$ is a linear combination of these solutions if

$$u_0 v_1 \neq u_1 v_0 . \qquad \text{(Linear independance)}$$

To see why this is true, assume that

$$\{w_n\}: w_0 \; w_1 \; w_2 \; w_3 \; \cdots$$

is another solution. Since $u_0 v_1 - u_1 v_0 \neq 0$, the system of linear equations

$$\begin{aligned} u_0 x + u_1 y &= w_0 \\ v_0 x + v_1 y &= w_1 \end{aligned}$$

has a unique solution. But this means that

$$w_0 \quad w_1 \quad w_2 \quad w_3 \quad \cdots$$

$$w_0 \downarrow u_0 x + v_0 y \qquad w_1 \downarrow u_1 x + v_1 y$$

$$\begin{aligned} w_2 \to \; & \alpha w_1 + \beta w_0 \\ &= \alpha(u_1 x + v_1 y) + \beta(u_0 x + v_0 y) \\ &= (\alpha u_1 + \beta u_0) x + (\alpha v_1 + \beta v_0) y \\ &= u_2 x + v_2 y \end{aligned}$$

$$\cdots \quad w_n = u_n x + v_n y .$$

This year you have not seen vector spaces; you will learn this concept next semester in the lecture Math 261. To set up the theory, we have to use some results of linear algebra. First, with the usual addition operation defined for sequences, together with multiplication with real numbers, set of sequences is a vector space ~~of~~ over the field of real numbers. Properties 1, 2 and 3 say that the set of solutions of (✩) is a vector sub-space. Property 4 means that the vector space of solutions of (✩) has dimension 2.

The main result of above observations is the following.

If $\{u_n\}$ and $\{v_n\}$ are two linearly independent solutions of a relation

$$a_n = \alpha a_{n-1} + \beta a_{n-2} \qquad (✩)$$

then any solution is a linear combination of $\{u_n\}$ and $\{v_n\}$. If the general terms of $\{u_n\}$ and $\{v_n\}$ are explicitly given by $u_n = f(n)$ and $v_n = g(n)$, then general solution is

$$w_n = A f(n) + B g(n)$$

for $A, B \in \mathbb{R}$.

Example 1 Given the relation

$$a_n = 5a_{n-1} - 6a_{n-2}. \qquad (1)$$

a) Let $\{a_n\}$ be a solution of (1) with $a_0 = 1$, $a_1 = 5$. Find a_5.

b) Show that the constant~~s~~ sequence $\{1\}$: $1\ 1\ 1\ \ldots$ is <u>not</u> a solution.

c) Show that $\{2^n\}$ is a solution.

d) Show that $\{3^n\}$ is a solution.

e) Show that $\{4^n\}$ is <u>not</u> a solution.

f) Find the general solution of (1).

g) Write the general term of the sequence in part a).

Solution

a) $a_0 = 1 \qquad a_1 = 5$

$$a_2 = 5\cdot a_1 - 6\cdot a_0 = 19$$
$$a_3 = 5a_2 - 6a_1 = 65$$
$$a_4 = 5a_3 - 6a_2 = 211$$
$$a_5 = 5a_4 - 6a_3 = 665.$$

b) $$\underset{u_2}{1} \neq 5\cdot\underset{u_1}{1} - 6\cdot\underset{u_0}{1} = -1 \qquad \{u_n\}: \underset{u_0}{1}\ \underset{u_1}{1}\ \underset{u_2}{1}\ \underset{u_3}{1}$$

$\Rightarrow$ $1\ 1\ 1\ 1\ \ldots$ is not a solution of (1).

c) $\{u_n\} = \{2^n\}$

$$5 \cdot \underbrace{2^{n-1}}_{u_{n-1}} - 6 \cdot \underbrace{2^{n-2}}_{u_{n-2}} = 2^{n-2}(\underbrace{5 \cdot 2 - 6}) = 2^{n-2} \cdot 4 = \underbrace{2^n}_{u_n}$$

$\Rightarrow \{u_n\} = \{2^n\}$ is a solution.

d)

$$5 \cdot \underbrace{3^{n-1}}_{u_{n-1}} - 6 \cdot \underbrace{3^{n-2}}_{u_{n-2}} = 3^{n-2}(5 \cdot 3 - 6) \qquad \{u_n\} = \{3^n\}$$

$$= 3^{n-2} \cdot 9$$

$$= 3^n = u_n \checkmark$$

$\{3^n\}$ is a solution.

e)

$$5 \cdot \underbrace{4^{n-1}}_{u_{n-1}} - 6 \cdot \underbrace{4^{n-2}}_{u_{n-2}} = 4^{n-2}(5 \cdot 4 - 6) \qquad \{u_n\} = \{4^n\}$$

$$= 4^{n-2} \cdot 14$$

$$\neq 4^n = u_n \quad \times$$

$\{4^n\}$ is <u>not</u> a solution.

f) From parts c) and d) we know two solutions of (1). These solutions are linearly independent:

$$\begin{array}{c|cc|cccc} & 0 & 1 & 2 & 3 & 4 & \\ \hline 2^n: & 1 & 2 & 4 & 8 & 16 & \cdots \\ 3^n: & 1 & 3 & 9 & 27 & 81 & \cdots \end{array}$$

$1 \cdot 3 - 1 \cdot 2 = 1 \neq 0.$

It follows that, any solution of (1) is of the form

$$a_n = A \cdot 2^n + B \cdot 3^n.$$

g) $\{a_n\}$: 1 5 19 65 211 665 ...

since any solution is of the form

$$a_n = A \cdot 2^n + B \cdot 3^n$$

we have to determine A and B.

For $n=0$ $\quad 2^0 A + 3^0 B = a_0 \Rightarrow A + B = 1$

For $n=1$ $\quad 2^1 A + 3^1 B = a_1 = 2A + 3B = 5$

Then $A = -2$ $\quad B = 3$ so

$$a_n = -(2)^{n+1} + 3^{n+1}.$$

check: $n=0$

$a_0 = -2 + 3 = 1$

$a_1 = -4 + 9 = 5$

$a_2 = -8 + 27 = 19$

$a_3 = -16 + 81 = 65$

$a_4 = -32 + 243 = 211$

$a_5 = -64 + 729 = 665$

$\vdots$

Characteristic Equation.

Up to now we have seen that, to write the general solution of a relation

$$\boxed{a_n = \alpha a_{n-1} + \beta a_{n-2}} \quad \text{--- } \boxed{(\star)}$$

it is sufficient to know two (independent) particular solutions.

In this part we learn how to find some particular solutions of $(\star)$. We wonder if there is any geometric progression (sequence) $\{u_n\} = \{r^n\}$ which is a solution of $(\star)$.

For the time being, we assume that r is real. Later on we will let it to be a complex number as well.

Now, r^n is a solution of $(\star)$ iff

$$r^n = \alpha r^{n-1} + \beta r^{n-2}$$

which can be written as

$$\boxed{r^2 - \alpha r - \beta = 0.} \quad \text{--- } \boxed{(C)}$$

Thus, we see that, roots of equation (C) provide us solutions of $(\star)$.

$\boxed{\text{(C) is called the characteristic equation of } (\star).}$

<u>Example 2.</u> Find the general solution of recursion

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

Characteristic equation of the relation is

$$r^2 - 5r + 6 = 0$$

which can be written as

$$(r-2)(r-3) = 0.$$

Thus, roots of characteristic equation are

$$r_1 = 2, \quad r_2 = 3.$$

It follows that $\{2^n\}$ and $\{3^n\}$ are (independent) solutions of the given relation and there is no other geometric progression which is a solution.

Note that in Example 1., we were given the sequences $\{2^n\}$ and $\{3^n\}$ and asked to show that they are two independent solutions. Then we could write the general solution. We see that, using the characteristic equation, we can obtain such solutions, (need not to wait for someone to give us some solutions.).

There may be the cases where the 'magic of char. eqn' does not work.

<u>Example 3.</u> Solve $a_n = 6a_{n-1} - 9a_{n-2}$.

Characteristic equation is

$$r^2 - 6r + 9 = (r-3)^2 = 0$$

and we can obtain only one solution $\{3^n\}$.

Now let the characteristic equation of (*) has only one root with multiplicity 2 (that is, a double root, a repeated root ...). This means that characteristic equation can be written in the form $(r-r_0)^2 = r^2 - 2r_0 + r_0^2$. for some $r_0 \in \mathbb{R}$. If we compare this form with (C):

$$r^2 - 2r_0 + r_0^2 = r^2 - \alpha r - \beta ,$$

we see that $\alpha = 2r_0$; $\beta = -r_0^2 \Rightarrow -\beta = \left(\frac{\alpha_0}{2}\right)^2$.

Then we conclude that char. eqn. of (*) has only one root (so gives only one solution) only when $\beta = -\frac{\alpha^2}{4}$. Now we will show that

in such a case, $\{r_0^n\}$ is a solution and also $\{n r_0^n\}$ is a solution.

Relation: $$a_n = \alpha a_{n-1} - \frac{\alpha^2}{4} a_{n-2} .$$

$\{r_0\}$: Char. eqn. $r^2 - \alpha r + \frac{\alpha^2}{4} = 0$

$$= \left(r - \frac{\alpha}{2}\right)^2 = 0$$

$$\Rightarrow r_0 = \frac{\alpha}{2} .$$

* $\{r_0^n\} = \left\{\left(\frac{\alpha}{2}\right)^n\right\}$ is a solution :

$$\alpha r_0^{n-1} - \frac{\alpha^2}{4} r_0^{n-2} = \alpha \cdot \left(\frac{\alpha}{2}\right)^{n-1} - \frac{\alpha^2}{4}\left(\frac{\alpha}{2}\right)^{n-2}$$

$$= \frac{\alpha^n}{2^{n-1}} - \frac{\alpha^n}{2^n}$$

$$= \frac{\alpha^n}{2^n}$$

$$= \left(\frac{\alpha}{2}\right)^n$$

$$= r_0^n \checkmark$$

right side of recursion

left hand side

* $\{n r_0^n\} = \{n(\frac{\alpha}{2})^n\}$ is a solution:

$$\alpha(n-1)r_0^{n-1} - \frac{\alpha^2}{4}(n-2)r_0^{n-2} = \alpha(n-1)\left(\frac{\alpha}{2}\right)^{n-1} - \frac{\alpha^2}{4}(n-2)\left(\frac{\alpha}{2}\right)^{n-2}$$

RHS

$$= (n-1)\frac{\alpha^n}{2^{n-1}} - (n-2)\frac{\alpha^n}{2^n}$$

$$= \left((2n-2)-(n-2)\right)\frac{\alpha^n}{2^n}$$

$$= n\frac{\alpha^n}{2^n}$$

$$= n r_0^n \quad \checkmark.$$

Then we observe that:

$$\boxed{\text{If } r_0 \text{ is a double root of (C), then } \{r_0\} \text{ and } \{n r_0\} \text{ are indep. solutions of (A).}}$$

Now, we can complete Example 3, by writing the general solution as

$$a_n = A\cdot 3^n + B\cdot n3^n.$$

Example 4 A sequence $\{u_n\}$ is defined as $u_0=1$, $u_1=5$ and $u_n = 2u_{n-1} - u_{n-2}$ for $n\geq 2$. Find the general term.

Characteristic equation is $r^2-2r+1=(r-1)^2=0$, thus we have the double root $r_0=1$. Then, general solution is $u_n = A\cdot 1^n + Bn1^n = A+Bn$.

$$\left.\begin{array}{ll} n=0 \Rightarrow & A = u_0 = 1 \\ n=1 \Rightarrow & A+B = u_1 = 5 \Rightarrow B=4 \end{array}\right\} \quad \boxed{u_n = 1+4n}.$$

__Example 5__ Let $b_0 = 2$, $b_1 = 5$ and for $n \geq 2$, b_n is equal to the arithmetic mean of the two preciding terms b_{n-1} and b_{n-2}:

b_0	b_1	b_2	b_3	b_4	b_5	
2	5	$7/2$	$17/4$	$31/8$	$65/16$	$\cdots$

Find the general term and compute $\lim_{n\to\infty} b_n$.

Recursion:

$$b_n = \frac{1}{2}(b_{n-1} + b_{n-2}).$$

Char. Eqn. $\quad 2r^2 - r - 1 = 0. \quad ((r-1)(2r+1) = 0)$

Roots $\quad r_1 = 1, \quad r_2 = -\frac{1}{2}.$

Gen. Soln: $\quad b_n = A + B\left(-\frac{1}{2}\right)^n.$

Using Init Cond.

$$\left.\begin{array}{lll} b_0 = 2 & \Rightarrow & A + B = 2 \\ b_1 = 5 & \Rightarrow & A - \frac{1}{2}B = 5 \end{array}\right\} \begin{array}{l} A = 4 \\ B = -2 \end{array}$$

$$\Rightarrow \quad b_n = 4 - 2\cdot\left(-\frac{1}{2}\right)^n$$

$$\Rightarrow \quad b_n = 4 + \left(-\frac{1}{2}\right)^{n-1}.$$

Fibonacci (Leonard da Pisa, 1170, 1250), played an important note in reviving ancient mathematics while making significant contributions of his own. He traveled to North Africa, Syria recognizing the advantages of the mathematical systems used in these countries. 'Liber Abaci', published in 1202 after his return to Italy, is based on bits of arithmetic and algebra that Leonardo had accumulated during his travels. The Liber Abaci introduced the Arabic place valued decimal system and the use of Arabic numerals (١٢٣٤٥٦٧٨٩ .) $\rightarrow$ (1 2 3 4 5 6 7 8 9) into Europe. In Liber Abaci the following, now famous, problem is posed. How many pair of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which become productive (mature) from the second month on.

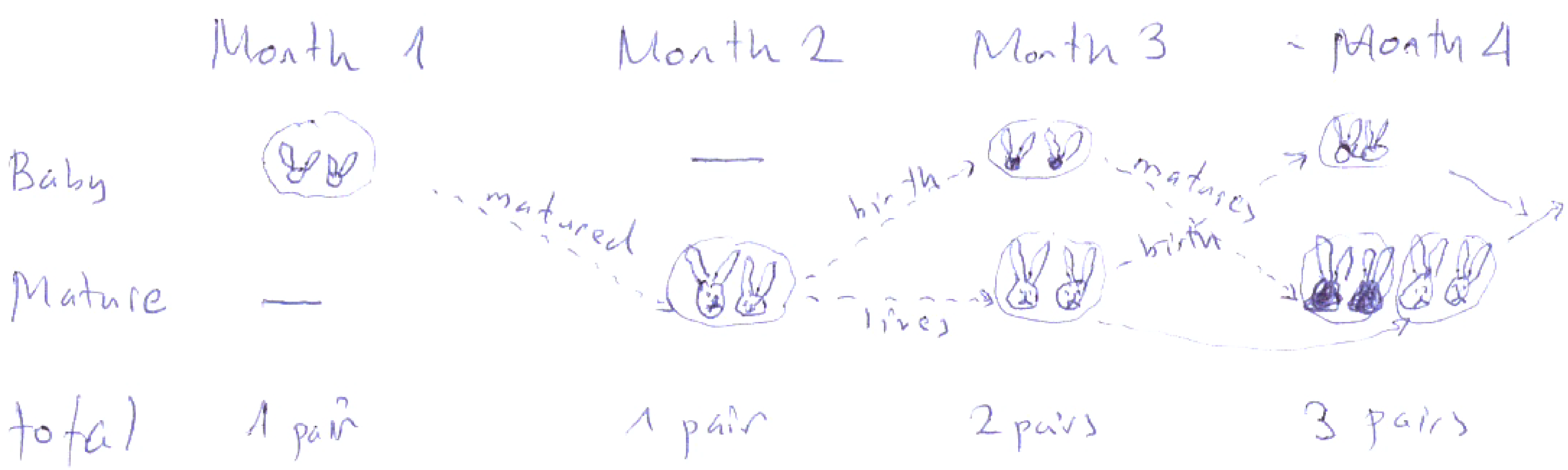
Example 6 [Fibonacci Rabbits] The number of rabbits, in a certain population, increases according to the following rules

- Each month, each mature female rabbit gives birth to a pair, one male and one female baby rabbits,
- Each new-born rabbit matures in one month,
- During the experiment period, no deaths are assumed (if a rabbit dies, an equivalent rabbit, from outside, is supplied).

If at the beginning (first day of first month) we start with a pair of baby rabbits, find an expression for the number of pairs at the n-th month.

Solution

At the beginning we have just a pair of baby rabbits (a total of one pair). At the second month, baby rabbits are grown up and we have a pair of mature rabbits (a total of one pair). At the third month, mature ~~of~~ rabbits of the previous month are still in the game and they have a pair of baby rabbits (a total of 2 pairs). We can observe the process schematically as follows.



We can simplify the scheme as

	M1	M2	M3	M4	M5	M6	M7	
Baby	1	0	1	2	3	5	8	
Mature	0	$\oplus$ 1	$\oplus$ 2	$\oplus$ 3	$\oplus$ 5	$\oplus$ 8	$\oplus$ 13	$\to \cdots$

In general the procedure is

	Month $n-2$	Month $n-1$	Month n
# Baby (b_n)	b_{n-2}	b_{n-1}	b_n
# Mature (m_n)	m_{n-2}	m_{n-1}	m_n
Total (f_n)	f_{n-2}	f_{n-1}	f_n.

(fibonacci)

Now we have

$$f_n = m_n + b_n$$
$$b_n = m_{n-1} = b_{n-2} + m_{n-2} = f_{n-2}$$
$$m_n = b_{n-1} + m_{n-1} = f_{n-1}$$

which implies that

$$f_n = f_{n-1} + f_{n-2} .$$

Although we have started the process from the first month, the sequence of fibonacci numbers $\{f_n\}$ is started with the index $n=0$. So, the number of rabbits in the first month is f_0. We have to solve the recursion

$$f_n = f_{n-1} + f_{n-2}$$

with the initial conditions $f_0 = f_1 = 1$.

First few terms of the sequence are

0	1	2	3	4	5	6	7	8	9	10	...
1	1	2	3	5	8	13	21	34	55	89	...

Characteristic equation of the relation is

$$r^2 - r - 1 = 0$$

with roots

$$r_1 = \frac{1-\sqrt{5}}{2} \qquad r_2 = \frac{1+\sqrt{5}}{2},$$

then the general solution is

$$f_n = A\left(\frac{1-\sqrt{5}}{2}\right)^n + B\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

To obtain the particular solution for $f_0 = f_1 = 1$ we have

f_0

for $n=0$ $\qquad A + B = 1$

f_1.

for $n=1$ $\qquad \left(\frac{1-\sqrt{5}}{2}\right)A + \left(\frac{1+\sqrt{5}}{2}\right)B = 1$

$$\frac{\overbrace{A+B}^{=1}}{2} + \frac{\sqrt{5}}{2}(-A+B) = 1$$

$$-A+B = \frac{1}{\sqrt{5}}$$

$$\Rightarrow \quad \begin{aligned} A + B &= 1 \\ -A + B &= \frac{1}{\sqrt{5}} \end{aligned} \quad \Rightarrow \quad \begin{aligned} B &= \frac{1+\sqrt{5}}{2\sqrt{5}} \\ A &= \frac{-1+\sqrt{5}}{2\sqrt{5}}. \end{aligned}$$

Then

$$f_n = -\left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n$$

or

$$f_n = \frac{1}{2\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \quad //.$$

(Note: The root $\varphi = \frac{1+\sqrt{5}}{2}$ of the char. eqn. is the golden ratio.)

The integer f_n is called a Fibonacci number. Just like $n!$, e_n, $\binom{n}{k}$; f_n is a special number which appear in many types of counting problems.

Example 7 Find general term of the sequence $\{u_n\}$ if $u_0 = u_1 = 1$ and $u_n = 2(u_{n-1} - u_{n-2})$.

Our sequence is

$$1 \quad 1 \quad 0 \quad -2 \quad -4 \quad -4 \quad 0 \quad 8 \quad +16 \quad 16 \quad 0 \quad \dots$$

Characteristic equation is

$$r^2 - 2r + 2 = 0.$$

The polynomial r^2-2r+2 is irreducible in the field $\mathbb{R}$ (has no factorization into linear factors). This means that it has no real roots and consequently, there isn't any $r \in \mathbb{R}$ such that $\{r^n\}$ is a solution of the given relation.

When we switch to the field of complex numbers, roots of gen. func. are

$$r_{1,2} = 1 \mp i$$

We couldn't fine real solutions, but we don't give up and proceed to find a solution through complex numbers. Thus we suggest a general solution of the form

$$u_n = A(1-i)^n + B(1+i)^n.$$

To determine the coefficients A and B, we proceed in the usual manner:

for $n=0$ $\qquad A + B = 1 \leftarrow u_0$

for $n=1$ $\qquad A(1-i) + B(1+i) = 1 \leftarrow u_1$

$$\underbrace{A+B}_{1} + i(-A+B) = 1$$

$$\Rightarrow \; -A+B=0$$

Then $A+B=1$, $-A+B=0 \Rightarrow A=B=1/2$.

General term of the sequence is

$$u_n = \frac{1}{2}\left[(1-i)^n + (1+i)^n\right]$$

It is remarkable that we obtained a general term containing some complex numbers for a sequence whose all terms are real. This means that for any integer n, the expression

$$(1-i)^n + (1+i)^n$$

is equal to a real number.

Example 8 Find general term of the sequence $\{v_n\}$ which satisfy the same recursion of the previous example with the initial terms $v_0 = 1$, $v_1 = 2$.

Since the general solution is obtained in the previous example we have to compute only the coefficients A and B in

$$v_n = A(1-i)^n + B(1+i)^n.$$

u_0

For $n=0$ $\qquad A + B = 1$

u_1

for $n=1$ $\qquad (1-i)A + (1+i)B = 2$

$$\underbrace{A+B}_{1} + i(-A+B) = 2$$

$$-A+B = \frac{1}{i} = -i$$

$$\Rightarrow \quad A - B = i$$

$$\Rightarrow \quad \left.\begin{aligned} A+B &= 1 \\ A-B &= i \end{aligned}\right\} \quad A = \frac{1+i}{2} \qquad B = \frac{1-i}{2} \quad \text{and}$$

$$v_n = \frac{1+i}{2}(1-i)^n + \left(\frac{1-i}{2}\right)(1+i)^n$$

or

$$v_n = (1-i)^{n-1} + (1+i)^{n-1}.$$

We see that, when the roots of ~~general~~ char. equ. are complex, then general term contains some complex quantities. Coefficients of the general term can be complex as well.

Now we have completed to examine second order relations. Our walkthrough for finding the solution is as follows.

Recursion

$$u_n = \alpha u_{n-1} + \beta u_{n-2}.$$

Write the characteristic equation:

$$r^2 - \alpha r - \beta = 0.$$

Find roots of char. eqn.

CASE 1 Roots are distinct $r_1 \neq r_2$.
(No matter whether roots are real or complex),
general solution is

$$u_n = A r_1^n + B r_2^n.$$

CASE 2 Roots are equal: $r_1 = r_2$.
(In this case roots are necessarily real).
General solution:

$$u_n = A r_1^n + B n r_1^n.$$

If there are initial terms given, then assigning particular values to n (in general $n=0$ and $n=1$), obtain a system of linear equations. Solve that system to determine the coefficients A and B.

Solution of Relations of order > 2.

General approach is quite similar to the case of order 2.

Given a relation of order t:

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_t u_{n-t}.$$

Characteristic equation is defined as

$$r^t - c_1 r^{t-1} - c_2 r^{t-2} - \cdots - c_t = 0.$$

Counting with multiplicities, characteristic equation has exactly t roots.

If r_0 is a simple root (not repeated), then $\{r_0^n\}$ is a solution of the recursion.

If r_0 is a root with multiplicity k (that is, characteristic equation has a factor $(r-r_0)^k$), then

$$\{r_0^n\}, \{n r_0^n\}, \{n^2 r_0^n\} \cdots \{n^{k-1} r^n\}$$

are all solutions of the relation.

It can be shown that the set of solutions constitute a vector space of dimension t. Thus, any solution of the relation is a linear combination of the solutions obtained from the roots of the characteristic equation.

If there are initial conditions given, then the corresponding coefficients can be obtained by writing t-equations in t-unknowns.

Example 9. Find general term of the sequence $\{a_n\}$ if $a_0 = 1$ $a_1 = 3$ $a_2 = 2$ $a_3 = 1$ $a_4 = 2$ and for $n \geq 5$

$$a_n = 2a_{n-1} + a_{n-4} - 2a_{n-5}.$$

Solution Characteristic equation is

$$r^5 - 2r^4 - r - 2 = 0.$$

Factorization of $r^5 - 2r^4 - r - 2$ is $(r-1)(r+1)(r^2+1)(r-2)$.

All roots are simple and

$$r_1 = 1 \quad r_2 = -1 \quad r_3 = -i \quad r_4 = i \quad r_5 = 2.$$

General solution is

$$a_n = A + B(-1)^n + C(-i)^n + D(i)^n + E\,2^n.$$

Respecting the initial conditions:

$$\begin{array}{lll}
n=0 & A + B + C + D + E = 1 & a_0 \\
n=1 & A - B - Ci + Di + 2E = 3 & a_1 \\
n=2 & A + B - C - D + 4E = 2 & a_2 \\
n=3 & A - B + Ci - Di + 8E = 1 & a_3 \\
n=4 & A + B + C + D + 16E = 2 & a_4
\end{array}$$

Solution is $A = \frac{9}{6}$ $B = -\frac{1}{6}$ $C = \frac{-1-3i}{5}$ $D = \frac{-1+3i}{5}$ $E = \frac{1}{15}$.

General term:

$$a_n = \frac{9}{6} - \frac{1}{6}(-1)^n - \frac{1+3i}{5}(-i)^n + \frac{-1+3i}{5}(i)^n + \frac{1}{15}\cdot 2^n.$$

Some Special Type of Non-Homogeneous Relations

Our theory works for homogeneous relations. Some particular type of non-homogeneous relations can be transformed into homegeneous relations by particular transformations. We consider two type of 'non-homogeneous' terms.

$\boxed{\text{CASE 1}}$

$$\underbrace{u_n = c_1 u_{n-1} + \cdots + c_t u_{n-t}}_{\text{homogeneous part}} + \underbrace{n^k}_{\text{non homogeneous part: a power of } n.}$$

To get rid of the non-homogeneous part we write the relation for $n+1$:

$$u_{n+1} = c_1 u_n + c_2 u_{n-1} + \cdots + c_t u_{n-t+1} + (n+1)^k$$

Now, difference of the relations is

$$u_{n+1} = (c_1+1)u_n + (c_2 - c_1)u_{n-1} + \cdots + (c_t - c_{t-1})u_{n-t+1} - c_t u_{n-t} + (n+1)^k - n^k.$$

Result is a new relation of order $t+1$ with non-homogeneous term $(n+1)^k - n^k$. Notice that $(n+1)^k - n^k$ is a polynomial in n of degree $k-1$.

If we apply the trick once more, we obtain a recursion of degree $t+2$ and non-homogeneous term of degree $k-2$. We can continue in this manner (for a total of $k+1$ times) to obtain a relation of order $n+k+1$ without any nonhomogeneous term.

Example 10 Find the general term of the sequence $\{u_n\}$ if $u_0=1$, $u_1=2$ and $u_n = u_{n-1} + 2u_{n-2} + n + 1$.

Solution.

$$\begin{array}{ll} n & u_n = u_{n-1} + 2u_{n-2} + n + 1 \\ n \to n+1 & u_{n+1} = u_n + 2u_{n-1} + n + 2 \end{array}$$

Difference

$$u_{n+1} = 2u_n + u_{n-1} - 2u_{n-2} + 1.$$

$$n \to n+1 \qquad u_{n+2} = 2u_{n+1} + u_n - 2u_{n-1} + 1$$

Difference

$$u_{n+2} = 3u_{n+1} - u_n - 3u_{n-1} + 2u_{n-2}.$$

Char. Eqn.

$$r^4 - 3r^3 + r^2 + 3r - 2 = 0$$

Factorization. $(r-2)(r+1)(r-1)^2 = 0$

Roots $r_1 = 2 \quad r_2 = -1 \quad r_3 = r_4 = 1.$

General Solution $u_n = A\,2^n + B(-1)^n + C + Dn$.

We have 4 unknowns and 2 initial conditions. But using the original recursion and given conditions, we can obtain as much 'initial terms' as we need.

$u_0 = 1 \quad u_1 = 2 \quad u_2 = u_1 + 2u_0 + 2 + 1 = 7 \quad u_3 = u_2 + 2u_1 + 3 + 1 = 15.$

$$\left.\begin{array}{ll} n=0 & A + B + C = 1 \\ n=1 & 2A - B + C + D = 2 \\ n=2 & 4A + B + C + 2D = 7 \\ n=3 & 8A - B + C + 3D = 15 \end{array}\right\} \quad \begin{array}{ll} A = 7/3 & B = 5/12 \\ C = -7/4 & D = -1/2 \end{array}$$

General term: $u_n = \frac{7}{3}\cdot 2^n + \frac{5}{12}(-1)^n - \frac{7}{4} - \frac{n}{2}$.

$\boxed{\text{CASE 2}}$

$$\underbrace{u_n = c_1 u_{n-1} + \dots + c_t u_{n-t}}_{\text{homogeneous part}} + \underbrace{\lambda^n}_{\text{non homogeneous term}}$$

($\lambda \in \mathbb{R}$, exponent n)

Here, our trick is the following:

Once write the original recursion by replacing n with $n+1$:

$$u_{n+1} = c_1 u_n + \dots + c_{n-t+1} + \lambda^{n+1}$$

Now, multiply all terms of the original relation by λ:

$$\lambda u_n = \lambda c_1 u_{n-1} + \dots + \lambda c_t u_{n-t} + \lambda^{n+1}.$$

Thus we have obtained two 'non-homogeneous' relations, one of order $t+1$ other of order t. Non-homogeneous terms of these relations are the same, so if we write the difference of these relations, the result will be a 'homogeneous' relation of order $t+1$.

Example 11 Given relation $u_n = 3u_{n-1} + u_{n-2} + 7^n$.

We first write

$$u_{n+1} = 3u_n + u_{n-1} + 7^{n+1}$$

then

$$7u_n = 3u_{n-1} + 7u_{n-2} + 7^{n+1}$$

Difference is

$$u_{n+1} = 10u_n - 2u_{n-1} - 7u_{n-2}.$$

Construction of Recursive Relations

In this part we will see examples of counting problems which can be solved by recursions.

Example 12 A frog, at the beginnig, sits at the origin (of real axis). At each time it jump to right either 1 unit or 2 units ahead. In how many different ways can this frog reaches to point 50?

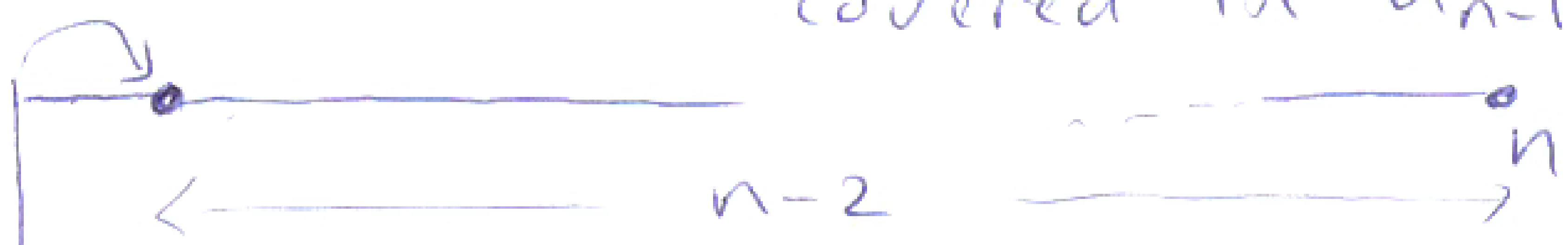
Solution. Let u_n be the number of ways the frog can reach to point n.

No matter how it goes to n, in the first jump there are two possibilities. It jumps to 1 or to 2.

If it jumps to 1:

The remaining part can be covered in u_{n-1} ways.

if it jumps to 2:



The remaining part can be covered in u_{n-2} ways.

$\Rightarrow \quad u_n = u_{n-1} + u_{n-2}$.

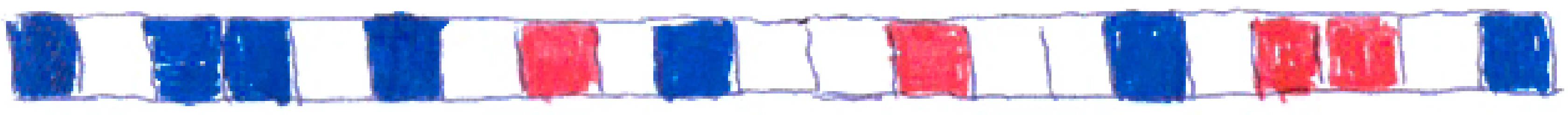
$u_1 = 1$ (only one way to jump to 1)

$u_2 = 2$ (two ways to reach to 2: $1+1$ or 2)

$\Rightarrow$ u_n is the standart fibonacci number $u_n = f_n$.

Example 13. There are n-boxes in a row. Find the number of ways of coloring these boxes such that

- each box is painted in red, blue or white.
- A blue box cannot be next to a red box.



Solution:

If there is no white box, then all the boxes must be either red or blue; there are two ways. Assume that there is at least one white box, and the first white box is at position $k \geq 1$

n-boxes $\longrightarrow$

$\underbrace{}_{k-1}$ k (first white box) $\underbrace{}$ $n-k$ boxes, colored in U_{n-k} ways.

no white here $\Rightarrow$ all blue or all red.

The number of ways, in this case: $2 \cdot U_{n-k}$

Since k takes any value $1, \dots, n$

$$U_n = \underset{\substack{k=1 \\ \text{(first box white)}}}{U_{n-1}} + \underset{k=2}{2U_{n-2}} + \dots + \underset{k=n-1}{2U_1} + \underset{\substack{k=n \\ \text{(last box white)}}}{2} + \underset{\substack{\text{no} \\ \text{white box.}}}{2}$$

for $n \to n+1$ $\quad U_{n+1} = U_n + 2U_{n-1} + 2U_{n-2} + \dots + 2U_1 + 2 + 2$

Difference of equations:

$$U_{n+1} = 2U_n + U_{n-1} \qquad \text{char eqn } r^2 - 2r - 1 = 0$$
$$\text{Roots: } 1 \mp \sqrt{2}$$

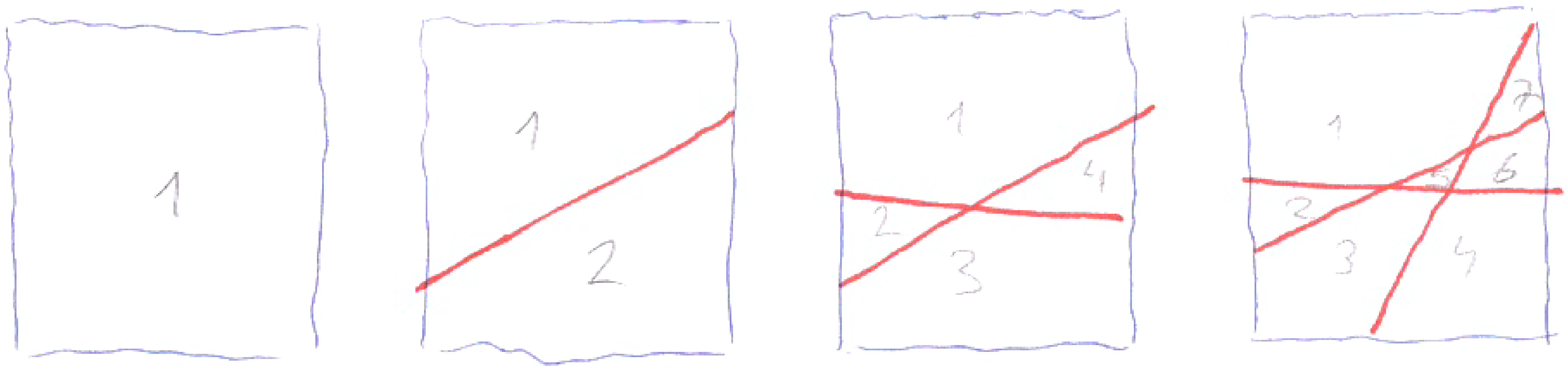
General Solution: $A(1-\sqrt{2})^n + B(1+\sqrt{2})^n$

Intial conditions: $U_1 = 3 \qquad U_2 = 7$ (all possible 9 ways $-$ $\begin{smallmatrix} BR \\ RB \end{smallmatrix}$).

$\Rightarrow A = \frac{-2+\sqrt{2}}{2}$
$\quad B = \frac{2+\sqrt{2}}{2}$ $\quad \Rightarrow \quad U_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}\right]$

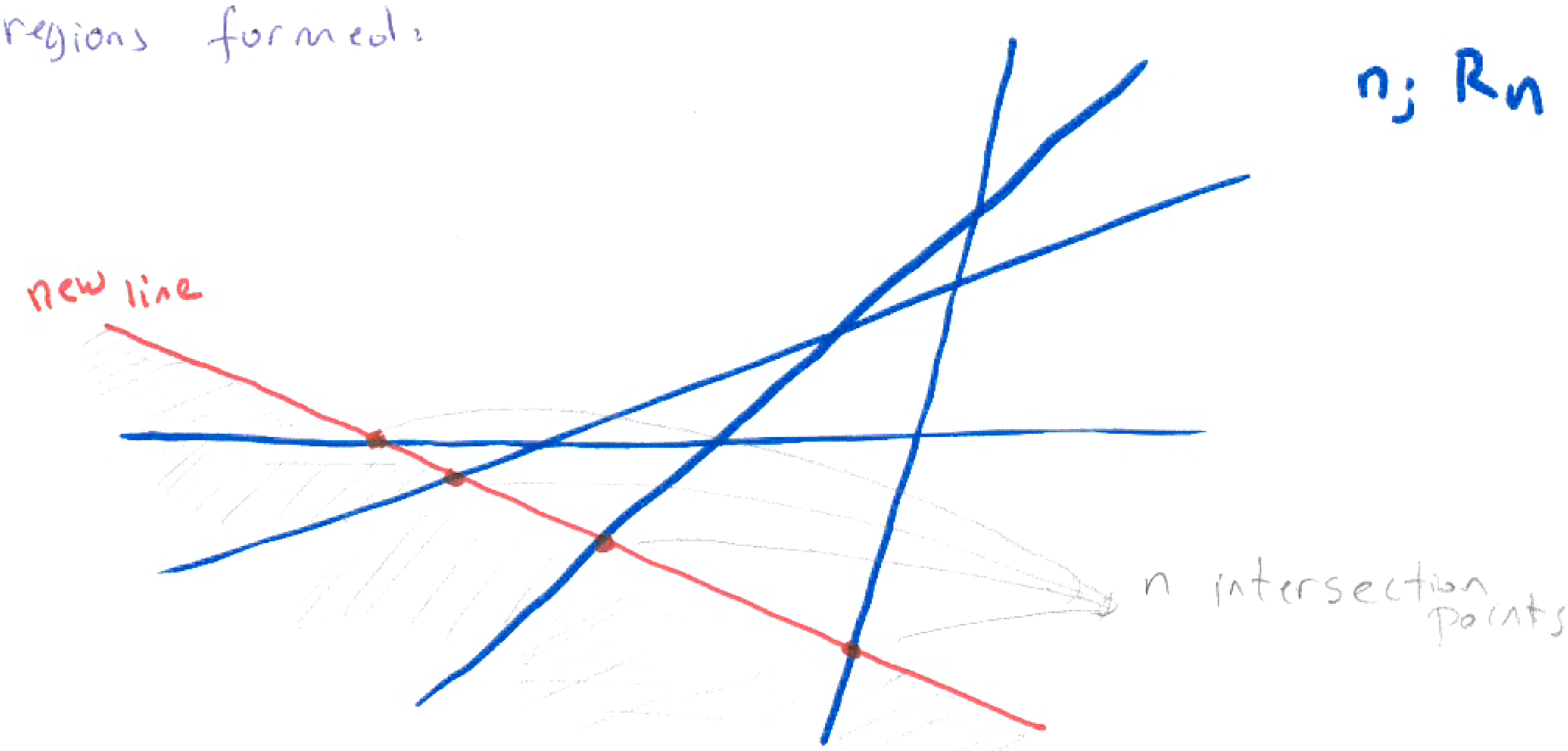
Example 14. Find the maximum possible of regions defined by n straight lines drawn in plane.

Solution Maximum possible number of regions is obtained when no line is parallel to any other and no set of three lines pass through a common point. Such a configuration is called 'n-lines in general position'. Let the number of regions in this case is R_n.

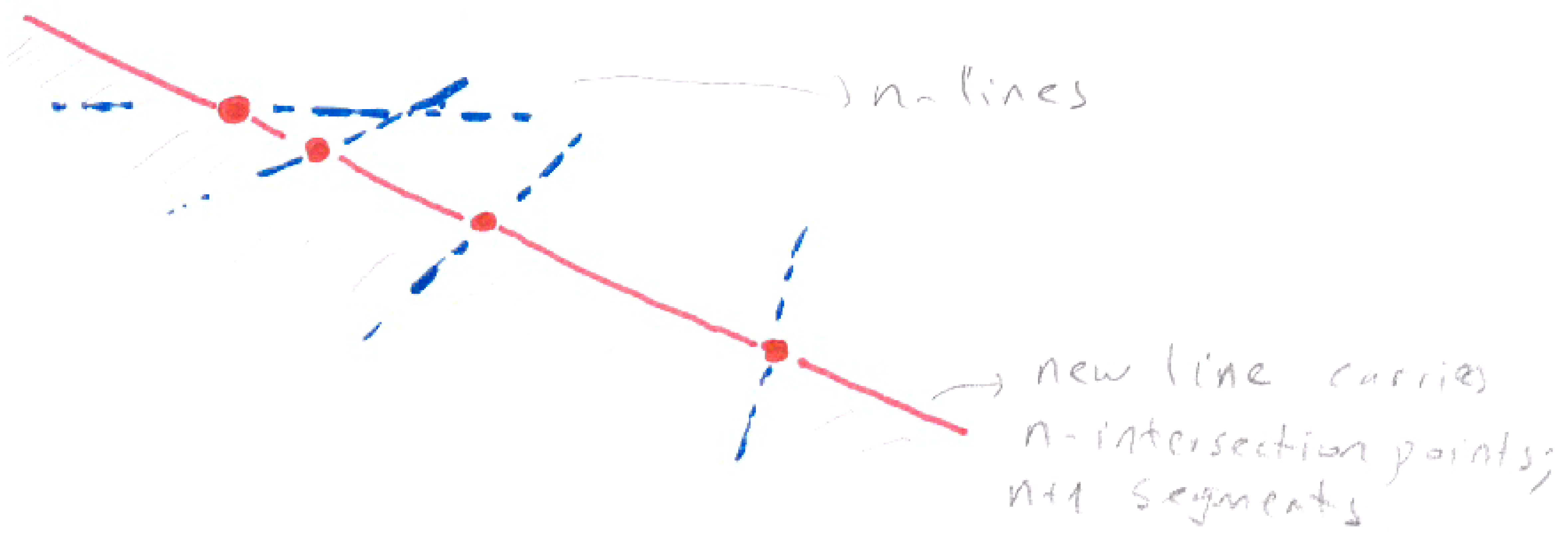


$n=0$ $R_0=1$ $n=1$ $R_0=2$ $n=2$ $R_0=4$ $n=3$ $R_0=7$

Now, assume there are n lines drawn and R_n regions formed:



When we draw a 'new line', it intersects all of the existing lines. So, there are n intersection points on this line, which separates the 'new line' into n+1 segments.



Each of these segments divides an already existing region into two parts. Thus, each segment corresponds to a new region. Then the number of all regions is the number of existing regions (R_n) plus the number of new regions ~~R_n~~ $(n+1)$. Hence

$$R_{n+1} = R_n + n + 1.$$

We need a homogeneous relation. Then

$$\begin{array}{ll} n \to n & R_{n+1} = R_n + n + 1 \\ n \to n+1 & R_{n+2} = R_{n+1} + n + 2 \end{array}$$

$$\begin{array}{ll} \text{Difference} & R_{n+2} = 2R_{n+1} - R_n + 1 \\ \text{again } n \to n+1 & R_{n+3} = 2R_{n+2} - R_{n+1} + 1 \end{array}$$

Once more Difference: $R_{n+3} = 3R_{n+2} - 3R_{n+1} + R_n$.

Now recursion is homogeneous and char eqn is

$$r^3 - 3r^2 + 3r - 1 = 0$$
$$\Rightarrow \quad (r-1)^3 = 0.$$

Since the roots are $r_1 = r_2 = r_3 = 1$, general solution is

$$R_n = A + Bn + Cn^2.$$

$$\left.\begin{array}{lll} n=0 & A = 1 & (R_0) \\ n=1 & A + B + C = 2 & (R_1) \\ n=2 & A + 2B + 4C = 4 & (R_2) \end{array}\right\} \quad A = 1 \quad \Rightarrow \quad u_n = \frac{1}{2}n(n+1) + 1.$$

$$\text{or} \quad u_n = \frac{n^2+n+2}{2}.$$