M E T U
Department of Mathematics

	Discrete Mathematics MidTerm I	
Code : *Math 112* Acad. Year : *2011-2012* Semester : *Spring* Instructor : *Finashin, Okutmuştur, Önal, Seven.* Date : *27.03.2012* Time : *17.40* Duration : *90 minutes*	Last Name : Name : Student No : Department : Signature :	**4** Questions on **4** Pages Total **60** Points

1	2	3	4		

READ THE PROBLEMS CAREFULLY AND GIVE DETAILED WORK

1. (15 pts) a) Find the number of monomials (monomial is a summand of the form $cx^iy^jz^k$, where c is a constant) in the expansion of $(x+y+z+1)^{100}$.

Solution: The number of monomials is equal to the number of integer triples (i,j,k), $i,j,k \geq 0$, $i+j+k \leq 100$. It is equal to the number of non-negative integer solutions of the equation $i+j+k+t = 100$, so, it is $\binom{103}{3} = \frac{100!}{100!3!}$.

b) Find the coefficient of x^8 in $(x^2+x+2)^7$.

Solution: Product $(x^2)^i x^j 2^k$, $i+j+k = 7$, appears with the coefficient $\frac{7!}{i!j!k!}$. Here, x^8 occurs if $2i+j = 8$, that is for the following triples (i,j,k): $(4,0,3)$, $(3,2,2)$, $(2,4,1)$, $(1,6,0)$. This gives

$$\frac{7!}{4!3!}2^3 + \frac{7!}{3!2!2!}2^2 + \frac{7!}{2!4!}2 + \frac{7!}{6!} = 35 \times 8 + 210 \times 4 + 105 \times 2 + 7 = 280 + 840 + 210 + 7 = 1337.$$

c) How many telephone numbers can be formed with two digits 1, two digits 2, and three digits 3, so that two (but not all three) digits 3 stand together ?

Solution: The first solution: $\frac{4!}{2!2!}$ permutations of $1,1,2,2$, then 5 possible places between 4 digits to insert 33 (together), and 4 places to insert 3 (separate). The result is $\frac{4!}{2!2!} \times 5 \times 4 = 120$.

The second solution: from $\frac{6!}{2!2!}$ permutations of $1,1,2,2,33,3$ we should subtract the ones with three digits 3 together, which were counted in two ways: as 33 and then 3, or 3 and then 33. The result is

$$\frac{6!}{2!2!} - 2\frac{5!}{2!2!} = 180 - 60 = 120.$$

The third solution: there are $\frac{7!}{2!2!3!}$ permutations of $1,1,2,2,3,3,3$, and among them $\frac{4!}{2!2!}\binom{5}{3}$ have no pair of digits 3 standing together, and $\frac{5!}{2!2!}$ have all three digits 3 standing together. The result is

$$\frac{7!}{2!2!3!} - \frac{4!}{2!2!}\binom{5}{3} - \frac{5!}{2!2!} = 210 - 60 - 30 = 120.$$

2. (15 pts) Suppose that a department contains 10 women and 6 men.

a) Three couples are married. In how many ways it could happen ?

Solution: Note that there are $C(6,3)$ ways to choose 3 men to be married, then there are $P(10,3)$ ways to choose three women to marry these three men. Thus the answer is:

$C(6,3)P(10,3)$

b) How many ways are there to form a committee with 5 members if it must have more women than men ?

Solution: Note that there are three distinct possibilities for the committee: there could be no men, or exactly one men or exactly two men. Thus the answer is:

$C(10,5)C(6,0) + C(10,4)C(6,1) + C(10,3)C(6,2)$

c) How many ways are there to arrange all 16 department members in a row, so that no two men stand next to each other ?

Solution: Note that for any ordering of the 10 women, there are 11 possible places to insert men; we choose 6. Thus there are $10!P(11,6)$ ways to arrange.

3. (15 pts) a) Four fair coins are tossed. Find the probability that at least three of them land heads up ?

Solution: There are five cases: {4H and 0T : 1/16}, { 3H and 1T : 4/16}, { 2H and 2T : 6/16}, {1H and 3T : 4/16}, { 0H and 4T : 1/16}

Thus the probability that at least three of them land heads up is $4/16 + 1/16 = 5/16$

b) Suppose we flip a coin four times. What is the probability that exactly three heads occur given that at least one head occurred ?

Solution: If we flip a coin four times we get $|S| = 16$. If we know at least one head occurs, define the new sample space as E, so we have $E = S - \{TTTT\}$, by which we get $|E| = 15$.

Let A : Have exactly 3 heads.

So $|A| = \frac{4!}{3!1!} = 4$.

Thus the answer is $\frac{|A|}{|E|} = 4/15$.

c) What is more probable: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled ?

Solution: I. Rolling two dice, to obtain 8 we have the following choices : $2-6, 3-5, 4-4, 5-3, 6-2$. Since the sample space has $6^2 = 36$ elements, the required probability is $5/36$.

(or once can do it by the following : $x_1 + x_2 = 8$ for $1 \leq x_i \leq 6,\ i = 1, \cdots, 6$ and let $x_i = y_i + 1$. Then we have $y_1 + y_2 = 6$ where $0 \leq y_i \leq 5$. We have $\binom{6+2-1}{6} = \binom{7}{6} = 7$ for $0 \leq y_i$. We have two more cases where $y_j = 6,\ j = 1, 2$; indeed $6 + 2$ and $2 + 6$. So the number of solutions is 7-2=5. Thus the required probability is 5/36.)

II. Rolling three dices, to obtain 8 we have : $x_1 + x_2 + x_3 = 8$ for $1 \leq x_i \leq 6,\ i = 1, \cdots, 6$. This implies $y_1 + y_2 + y_3 = 5$ where $x_i = y_i + 1$ for each $i = 1, \cdots, 6$. So we have $\binom{5+3-1}{5} = \binom{7}{5} = 21$ solutions. Thus the required probability is $21/216$.

(or one can check each cases : (1,1,6); (1,2,5) ;...; (1,6,1); (2,1,5); (2,2,4);...;(2,5,1); (3,1,4);(3,2,3);(3,3,2);(3,4,1); (4,1,3);(4,2,4);(4,3,1);(5,1,2);(5,2,1);(6,1,1) We have $6 + 5 + 4 + 3 + 2 + 1 = 21$ cases. Thus the required probability is 21/216)

Obviously $5/36 > 21/216$. Hence the first case is more probable.

4. (15 pts) Two distinct integers $1 \le a, b \le 100$ are chosen at random.

a) What is the probability that one of them is not greater than 10 ? Estimate whether the probability is less or more than 20% ?

Solution: $p(E) = \frac{|E|}{|S|}$, $\quad |S| = \binom{100}{2}$, $\quad$ event E: $a \le 10$, or $b \le 10$.

$\overline{E}$: $10 < a, b \le 100$, $\quad |\overline{E}| = \binom{90}{2}$, $\quad |E| = \binom{100}{2} - \binom{90}{2}$.

$p(E) = 1 - \frac{90 \times 89}{100 \times 99}$, $\quad$ where $\frac{90 \times 89}{100 \times 99} = \frac{89}{110} > \frac{88}{110} = \frac{8}{10}$.

Thus, $p(E) < 1 - \frac{8}{10} = \frac{2}{10} = 20\%$.

b) What is the probability that one of these two numbers is divisible by 3, or 5 ?

Solution: Among $1, 2 \ldots, 100$ there are

$\lfloor \frac{100}{3} \rfloor = 33$ numbers divisible by 3,
$\lfloor \frac{100}{5} \rfloor = 20$ divisible by 5,
$\lfloor \frac{100}{15} \rfloor = 6$ divisible by both.

Thus, $33 + 20 - 6 = 47$ are divisible by 3 or 5, and 53 are not divisible neither by 3, nor by 5.

Event E: a is divisible by 3 or 5, or b is divisible by 3 or 5
$\overline{E}$: both a and b are not divisible neither by 3, nor by 5,

$|\overline{E}| = \binom{53}{2}$, $\quad |E| = \binom{100}{2} - \binom{53}{2}$, and thus, $\quad p(E) = \frac{|E|}{|S|} = 1 - \frac{53 \times 52}{100 \times 99}$.