METU Department of Mathematics

Math 112

Discrete Mathematics

INDEPENDENT EVENTS - CONDITIONAL PROBABILITY

(Week 12)

LECTURE NOTES

Ali Doğanaksoy
April 2020

INDEPENDENT EVENTS / CONDITIONAL PROBABILITY

Consider the experiment of tossing a fair die and flipping a fair coin at the same time and define events O and T as

- Event O : The die faces a 1,

- Event T : The coin faces 'Tails'.

Before reading further, can you decide (at least intiutively) whether events O and T are dependent or independent.

Now consider another experiment, choosing a student from a classroom and let the events B and Y be defined as

- Event B : Chosen student has black hair,

- Event Y : Chosen student has yellow hair.

What is your intuition about these events? Are B and Y dependent or independent?

Third experiment is choosing a student of our university and we define the events M and G as

- Event M : Chosen student is from Mathematics Department,
- Event G : Chosen student is a graduate (M.Sc. or Ph.D.) student.

What about M and G? Are these event dependent or not?

Since the coin to face up 'Heads' or 'Tails' has no effect on the die to face up a '1', it is quite natural to say that O and T are indpendent

Events B and Y are disjoint events, that is, they cannot happen at the same time. Either one of them may happen or the other or none. In particular, if one of these events occurs, certainly the other can't happen. So we tend to say that these events are dependent.

For the events M and G, either response 'Dependent' or 'Independent' is incorrect. We can't decide whether these events are dependent or independent. Dependence of these events depends on the distribution of students in the university.

In the first case we say that O does not depend on T because, probability of event O is 1/6 whether the coin faces up 'Tails' or not. In particular, knowing that event T has happened has no effect on the probability of O.

On the other hand, in the second case, probability of event B is ratio of the number of black haired students to the number of all students in the class. If we are said that event Y has happened, then probability of M reduces to zero. Knowing that event Y has happened has an effect on the probability of B. So we conclude that event M is dependent with Y.

For the last case, assume that the number of students in the university is distributed as follows:

	Mathematics	Others	Total
Undergraduate	500	19,500	20,000
Graduate	175	6,825	7,000
Total	675	26,325	27,000

Probability of event M is (Number of maths students)/(Number of all students), so probability(M) = 675/27000 = 1/40

If we are given that the chosen student is a graduate student, then in this case probability (M if G happened) = (number of maths grad)/(number of all grad), so that probability(M if G happened) = 175/7000 = 1/40.

We see that probability of a chosen student to be from mathematics department is 1/40 wheteher we know or don't know that the student is a graduate student. So, probability of the event M seems to be independent of G.

Now, consider the same case with the folowing figures:

	Mathematics	Others	Total
Undergraduate	460	19,540	20,000
Graduate	140	6,860	7,000
Total	600	26,400	27,000

In this case, probability(M) = 600/27000 = 1/45 and

probability(M if G happened) = 140/7000 = 1/50.

We see that when we are informed that the student is a graduate student, probability of the event M reduces from 1/45 to 1/50. Since the probability of the event M is not the same when we do not have any information about G and when we know that G has happened, we feel that G is <u>not</u> independent of M.

Above discussion shows that independence of two events is not a matter how these events are defined, rather, independence is a concept related with the probabilities of events. A pair of events may be independent for a certain distribution, whereas they *turn out to be dependent for some other distribution. The concept of independence we are interested here is **independence in the context of probability.***

Given two events A and B, if the probability of event A remains the same when we are given the information that event B has happened, then we say that the event A is independent of B.

After introducing the definitions and the basic facts, we will show that if A is independent of B, then B is also independent of A. Thus, independence relation is symmetric and, we say that A and B are independent events to mean that A is independent of B and B is independent of A.

*The notation Pr(A) stands for the probability of A. The notation Pr(A|B) is used to mean the probability of A when it is known that B has certainly happened. As Pr(A|B) is probability of A under a certain condition, this notion is called **conditional probability.***

Pr(A|B)=Probability the event A to happen when we know that event B happened.

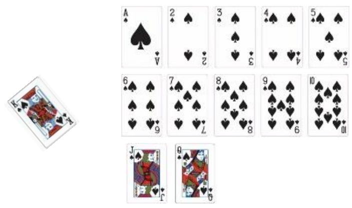
So with this notation, our definition of independence is

A and B are independent if and only if Pr(A) = Pr(A|B).

Example 1. A standart deck of playing cards consists of 52 cards. There are four suits (♠♥♣♦), and 13 cards of each suit are ranked as A,2,3,4,5,6,7,8,9,10,J,Q,K. Cards with ranks 'J, Q or K' are called face cards.

From a deck of playing cards when we choose a card randomly, probabilty of this card to be a king (K) is 4/52, $Pr(K) = 1/13$.

When we are given that the chosen card is a spades (♠), as there are 13 spades cards only one of which is a king, $Pr(K\,/\,♠) = 1/13$. So we observe that



$$Pr(K) = Pr(K\,/\,♠)$$

which implies that

Events 'K' and ♠' are independent.

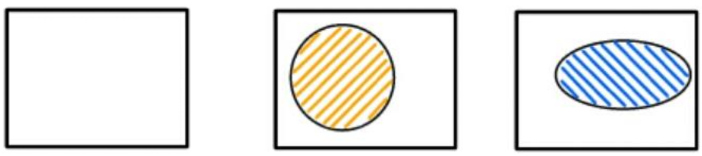
When we are given that the card is a face card, since there are 12 face cards, 4 of which are K's, we have $Pr(K\,/\,Face) = 4/12 = 1/3$. Thus

$$Pr(K) \neq Pr(K\,/\,Face)$$

so

Events 'K' and 'Face Card' are dependent.

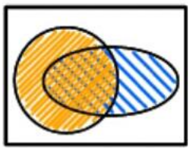
Let the sample space of an experiment be S and assume that we are interested with events A and B.



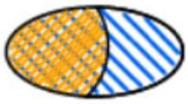
Sample Space S Set A Set B

Probability of A, Pr(A) is by definition, the number of points (simple events) in A divided by the number of points in the sample space.

When it is given that the event B has happened, and we are asked to find the probability of A, we focus on the set B rather than the universal set S and we try to find out how many points that belong to B also belong to A. But this leads us to the definition of the intersection of A and B.



In turn, we are interested in the points which are contained in B and particularly in those which are also contained in A. As the points of sample space S which are not in B are out of our interest, our new sample space is B, and we have to find the ratio of the number of points in the intersection to the number of points in B.



Finally we get the formula which is known as ***Bayes' Theorem:***

$$Pr(A/B) = Pr(A \cap B)/Pr(B).$$

THEOREM. *Two events A and B are independent if and only if*

$$Pr(A \cap B) = Pr(A) \cdot Pr(B).$$

Proof. *Events A and B are independent if and only if* $Pr(A) = Pr(A/B) = Pr(A \cap B)/Pr(B)$ *which is equivalent to* $Pr(A) \cdot Pr(B) = Pr(A \cap B)$. *QED*

COROLLARY 1. *If an event A is independent of B, then B is independent of A.*

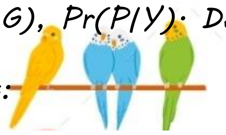
Proof. $Pr(A) = Pr(A/B) \Leftrightarrow Pr(A) = Pr(A \cap B)/Pr(B) \Leftrightarrow Pr(B) = Pr(A \cap B)/Pr(A)$, *but the last equality is equivalent with* $Pr(B) = Pr(B/A)$. *QED*

COROLLARY 2. For any events A and B following equality holds.

$$Pr(A|B) \cdot Pr(B) = Pr(B|A) \cdot Pr(A)$$

Proof. $Pr(A|B) \cdot Pr(B) = Pr(A \cap B) = Pr(B \cap A) = Pr(B|A) \cdot Pr(A)$. QED

Example 2. In a petshop there are 24 parrots (6 blue, 9 green and 9 yellow) and 48 lovebirds (12 blue, 15 green and 21 yellow). For a randomly picked bird, compute following probabilities: $Pr(B|P)$, $Pr(P|B)$, $Pr(P|G)$, $Pr(P|Y)$. Determine whether the following pairs of events are independent or not:



- P and B,
- P and G,
- P and Y.

(Events are denoted by their initial letters Parrot → P, Yellow → Y, ⋯ .)

	Parrots	Lovebirds	Total
Blue	6	12	**18**
Green	9	15	**24**
Yellow	9	21	**30**
Total	**24**	**48**	**72**

Solution.

Since $Pr(B \cap P) = 6/72$ and $Pr(P) = 24/72$, $Pr(B|P) = 6/24 = 1/4$.

Since $Pr(P \cap B) = 6/72$ and $Pr(B) = 18/72$, $Pr(P|B) = 6/18 = 1/3$.

Since $Pr(P \cap G) = 9/72$ and $Pr(G) = 24/72$, $Pr(P|G) = 9/24 = 3/8$.

Since $Pr(P \cap Y) = 9/72$ and $Pr(Y) = 30/72$, $Pr(P|Y) = 9/30 = 3/10$.

$Pr(P) = 24/72 = 1/3$, then

$Pr(P) = Pr(P|B) \Rightarrow$ 'Choosen bird is a parrot' and 'choosen bird is blue' are <u>independent</u>.

$Pr(P) \neq Pr(P|G) \Rightarrow$ 'Choosen bird is a parrot' and 'choosen bird is green' are <u>dependent</u>.

$Pr(P) \neq Pr(P|Y) \Rightarrow$ 'Choosen bird is a parrot' and 'choosen bird is yellow' are <u>dependent</u>.

Example 3. There are three boxes, labeled A, B and C. In A there are two silver coins, in B there are two gold coins and in C there is a gold and a silver coin. You choose one of the boxes randomly (without seeing its label) and without looking at its contents you pick one of the coins from the box. If the chosen coin is silver, what is the probability that the other coin in the chosen box is gold?

Solution. We are asked to compute $Pr(C|G)$.

There are exactly two possible ways of choosing a gold coin:

- Box B is chosen and a coin is picked $Pr(B \cap G) = (1/3)(1) = 1/3$,
- Box C is chosen and the gold coin is picked $Pr(C \cap G) = (1/3)(1/2) = 1/6$.

Then $Pr(G)=(1/3)+(1/6)=1/2$.

Now, $Pr(G|C) = 1/2$ (when the chosen box is C, probability of picking a gold coin).

Then $Pr(G \cap C)= Pr(G|C) \cdot Pr(C)=(1/2)(1/3)=1/6$.

It follows that $Pr(C|G)= Pr(C \cap G)/Pr(G) = (1/6)/(1/2) = 1/3$.

Example 4. *If a mother says that 'Zeynep is the elder of my two children' what is the probability that the other child is also a girl?*

Solution. For a family with two children, sample space of possibilities is {BB,BG,GB,GG} (GB means elder one is a Girl; other one is a Boy…). The statement of the mother reduces the space of possibilites to {GB,GG}. In only one of two cases of the sample space both children are girls (event GG). So probability that the other children is a girl is 1/2.

Example 5. *If a mother says that 'Zeynep is one of my two children' what is the probability that the other child is also a girl?*

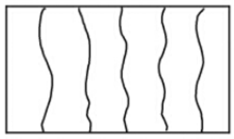
Solution. Sample space of possibilities is now {GG,BG,GB}. Then the probablity that the other child is a girl (event GG) is 1/3.

Exercise. *Formulate solutions of Example 5 and Example 6 in terms of conditional probabilities.*

Let the sample space of an experiment S be a disjoint union of subsets $A_1, A_2, \cdots, A_n$, that is, the given subsets are mutually pairwise disjoint and

$$S = A_1 \cup A_2 \cup \cdots \cup A_n.$$

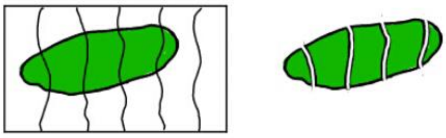
S



$S = A_1 \cup A_2 \cup \cdots \cup A_n$

Any event $X \subset S$ can be writen as

$$\begin{aligned} X &= X \cap S \\ &= X \cap (A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= (X \cap A_1) \cup (X \cap A_2) \cup \cdots \cup (X \cap A_n) \end{aligned}$$



and since $X \cap A_1$, $X \cap A_2$, $\cdots$, $X \cap A_n$ are pairwise disjoint sets we obtain

$$Pr(X) = Pr(X \cap A_1) + Pr(X \cap A_2) + \cdots + Pr(X \cap A_n).$$

For each $i = 1, \cdots, n$ we can write $Pr(X \cap A_i) = Pr(X \mid A_i) \cdot Pr(A_i)$ to get

$$Pr(X) = Pr(X|A_1)Pr(A_1) + Pr(X|A_2)Pr(A_2) + \cdots + Pr(X|A_n)Pr(A_n)$$

If we are given the conditional probabilities $Pr(X \mid A_i)$, $i = 1, \cdots, n$ and the probabilities $Pr(A_i)$, $i = 1, \cdots, n$ then

$$\boxed{Pr(A_i \mid X) = Pr(Pr(X \mid A_i) \cdot Pr(A_i) \;/\; [Pr(X \mid A_1)Pr(A_1) + \cdots + Pr(X \mid A_n)Pr(A_n)]}$$

for $i = 1, \cdots, n$.

Example 6. In a factory, there are 3 production lines A, B and C. This lines produce 10,000, 18,000 and 2,500 pieces a day, respectiely. Of all pieces produced in line A, 0.2% are defective. Percentage of defective pieces of productions of lines B and C are 0.15% and 0.4%, respectively.

At the end of the day a piece is picked (out of the entire production) randomly and seen to be defective. What is the probability that it is produced in A?

Solution. Total production is 30,500 pieces a day. So probabilities of a randomly chosen piece to be produced in A, B and C are

Pr(A) = 100/305, Pr(B) = 180/305 and Pr(C) = 25/305.

We are given that

Pr(D|A) = 20/10000, Pr(D|B) = 15/10000 and Pr(D|C) = 40/10000

where D is the event tahat a randomly chosen piece is defective. It follows that

Pr(D) = Pr(D|A)Pr(A) + Pr(D|A)Pr(A) + Pr(D|A)Pr(A)

= (20/10000)(100/305) + (15/10000)(180/305) + (40/10000)(25/305)

= 5700/3050000

= 0.001869… .

So, approximately 0.1869% of entire production is defective. Now we compute

Pr(A|D) = Pr(D|A).Pr(A) / Pr(D)

= (20/10000)(100/205) / 0,001869

≈ 0,35.

This means that ~35% of all defective pieces are produced in the line A.

Exercise. In Example 6 Show that Pr(B|D) ≈ 0.47 and Pr(C|D) ≈ 0.18.

Example 7. A cab was involved in a hit and run accident at night. Two cab companies, the Green and the Blue, operate in the city. 85% of the cabs in the city are Green and 15% are Blue. A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colours 80% of the time and failed 20% of the time.

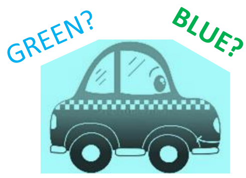
What is the probability that the cab involved in the accident was Blue rather than Green knowing that this witness identified it as Blue?

(Question is taken from: https://magesblog.com/post/2014-07-29-hit-and-run-think-bayes/)

Solution. First we list the probabilities we are given in the statement of the problem:

Pr(Car is green) = .85

Pr(Car is blue) = .15



Pr(Witness reports blue / Car is blue) = 0.8

Pr(Witness reports green / Car is blue) = 0.2

Pr(Witness reports blue / Car is green) = 0.8

Pr(Witness reports green / Car is green) = 0.2

There are exactly two possibilities for the witness to report a blue car,

- The car is actually blue and witness reports is correctly,
- The car is actually green and witness reports it incorrectly.

So we get

Pr(Witness reports Blue) = Pr(Witness reports blue / Car is blue).Pr(Car is Blue) + Pr(Witness reports blue / Car is green).Pr(Car is green)

= (0.8)(0.15)+(0.2)(0.85)

= 0.29.

From Corollary 2, it follows that

$$\text{Pr(Car is Blue/Witness reports blue)} = \frac{\text{Pr(Witness reports blue/Car is Blue) Pr(Car is Blue)}}{\text{Pr(Witness reports blue)}}$$

Finally,

$$\text{Pr(Car is Blue / Witness reports blue)} = (0.2)(0.15) / (0.29) = 0.4138.$$

Color of the car is more likely green even though the witness reports it to be blue.

Exercise. Compute the probability of the event that the car actually is green when the witness reports it to be green.

Example 8. In a box, there are 1 biased and 99 fair coins, where the biased coin faces 'Tails' with probability 0.75. You pick a coin randomly and flip it. If it faces 'Tails', what is the probability that it is biased?

Solution. We have the probabilities

Pr(Fair) = 0.99, and Pr(Biased) = 0.01,

Pr(Tails / Fair) = Pr(Tails / Fair) = 0.5,

Pr(Tails / Biased) = 0.75 and Pr(Tails / Biased) = 0.25.

To observe the coin faces 'Tails', there are just two possibilities:

- Coin is biased and it faces tails,

  Probability of this event is $Pr(Tails \cap Biased) = Pr(Tails \mid Biased) \cdot Pr(Biased)$

  $= (0.75)(0.01) = 0.0075.$

- Coin is fair and it faces tails,

  Probability of this event is $Pr(Tails \cap Fair) = Pr(Tails \mid Fair) \cdot Pr(Fair)$

  $= (0.5)(0.99) = 0.495.$

So we obtain Pr(Tails) = 0.0075 + 0.495 = 0.5025.

We use Corollary 2 to write

$$Pr(Biased \mid Tails) = Pr(Tails \mid Biased) \cdot Pr(Biased) \; / \; Pr(Tails).$$

The we obtain

$$Pr(Biased \mid Tails) = (0.75)(0.01) \; / \; 0.5025) = 0.0149.$$

Example 9. You repeat the experiment in Example 8, but now you flip the coin 3 times and you observe that it faces 'Tails' at all three flips. What is the probability that the coin is really biased?

Solution. Our probabilities are

$Pr(Fair) = 0.99$, and $Pr(Biased) = 0.01$,

$Pr(3Tails \mid Fair) = (0.5)^3 = 0.125$,

$Pr(3Tails \mid Biased) = (0.75)^3 = 0.421875$.

With the same reasoning in the perevious example,

$$Pr(3Tails) = Pr(3Tails \mid Biased) \cdot Pr(Biased) + Pr(3Tails \mid Fair) \cdot Pr(Fair)$$

$$= (0.421875)(0.01) + (0.125)(0.99) = 0.12793.$$

So

$$\begin{aligned} Pr(Biased \mid 3Tails) &= Pr(3Tails \mid Biased) \cdot Pr(Biased) \; / \; Pr(3Tails) \\ &= (0.421875)(0.01) \; / \; (0.12793) \\ &= 0.032977\cdots . \end{aligned}$$

If the coin shows 'Tails' face in all of three flippings, then it is biased with probability 3.3%.

Exercise. Consider the experiment of Example 9. But you flip the coin k times and you conclude that the coin is biased if it faces 'Tails' at all flippings. Find the smallest value of k so that when you conclude that the coins is biased, probability that the coin is actually biased is not less than 90%.

Example 10. *A binary medical test is a method which results either positive or negative. Positive result indicates the presence of a condition, such as a disease or infection, and negative result means that the condition is not present. For such a test there are two types of errors: a* ***false positive*** *is an error in which the test results (improperly) positive, although the condition is not present. A* ***false negative*** *is an error where the result is (improperly) negative when the condition actually is present.*

In a country with population 80.000.000, it is estimated that 625.000 people are infected. A laboratory developed a test and experimented on a large scale. Experiments showed that when applied to 10.000 healthy (not infected) people, the test resulted positive in only 20 cases. When the test is experimented on 5.000 defected people result is negative only in 25 cases. Is this test reliable?

Solution. For deciding on reliability of the test, we have to compute false positive and false negative probabilities.

Our immediate probability values are

$$Pr(Infected) = 625/80000 = 1/128000,$$

$$Pr(Not\ Infected) = 127999/128000,$$

$$Pr(Test\ Positive\ /\ Not\ Infected) = 20/10000 = 1/500,$$

$$Pr(Test\ Negative\ /\ Not\ Infected) = 499/500,$$

$$Pr(Test\ Negative\ /\ Infected) = 25/5000 = 1/200,$$

$$Pr(Test\ Positive\ /\ Infected) = 199/200.$$

We have to compute

$$Pr(False\ Negative) = Pr(Infected\ /\ Test\ Negative).$$

and

$$Pr(False\ Positive) = Pr(Not\ Infected\ /\ Test\ Positive)$$

$$\Pr(\text{Test Positive}) = \Pr(\text{Test Positive} \mid \text{Not Infected})\Pr(\text{Not Infected}) + \Pr(\text{Test Positive} \mid \text{Infected})\Pr(\text{Infected})$$

$$= (1/500)(127999/128000) + (199/200)(1/128000)$$

$$= 0{,}00976\cdots.$$

$$\Pr(\text{Test Negative}) = \Pr(\text{Test Negative} \mid \text{Not Infected})\Pr(\text{Not Infected}) + \Pr(\text{Test Negative} \mid \text{Infected})\Pr(\text{Infected})$$

$$= (499/500)(\ 127999/128000)+(1/200)(1/128000)$$

$$= 0{,}990242\cdots.$$

Then

$$\Pr(\text{False Negative}) = \Pr(\text{Infected} \mid \text{Test Negative})$$

$$= \Pr(\text{Test Negative} \mid \text{Infected})\Pr(\text{Infected}) \,/\, \Pr(\text{Test Negative})$$

$$= (1/500)(1/128000)\ /(0{,}990242)$$

$$= 0{,}000039.$$

$$\Pr(\text{False Positive}) = \Pr(\text{Not Infected} \mid \text{Test Positive})$$

$$= \Pr(\text{Test Positive} \mid \text{Not Infected})\Pr(\text{Not Infected}) \,/\, \Pr(\text{Test Positive})$$

$$= ()(\ 0.9922) \,/\, (0{,}00976)$$

$$= 0{,}7966$$

False negative probability is very low. When the test results negative, accuracy is very high, namely the result is correct by 99.99%

On the other hand, false positive probability is very high: positive test result accuracy is less than 21%. This means that out of 100 people with a positive result, 79 are not actually infected.

We conclude that when test results negative, body is not infected for certain (no quarantine, no treatment required). But, when the test is positive, no conclusion derives, (out of 100 positive results, actually about 80 are disinfected). Positive result is unreliable and it cannot be the basis of a decision to begin a treatment procedure.