List No:

METU
DEPARTMENT OF MATHEMATICS

2014-2015 Fall Semester
Math 115 Analytic Geometry
Final Examination
Monday, January 10th, 2015

Name, Last Name:	
Student No:	
Department:	Signature:

Duration: 115 minutes - 7 Questions on 4 Pages - Total 80 Points

QUESTION 1 (10 points)

Given the straight lines

$$\ell_1: \begin{cases} x = 2t+3 \\ y = t+4 \end{cases} \quad \text{and} \quad \ell_2: \begin{cases} x = 2t+3 \\ y = 4t+4 \end{cases}.$$

Write an equation of the straight line which passes through $\ell_1 \cap \ell_2$ and which bisects the angle between ℓ_1 and ℓ_2.

Obviously the point $P(3,4)$ is the intersection point of ℓ_1 and ℓ_2. We can take $\vec{u} = (2,1)$ and $\vec{v} = (1,2)$ as direction vectors of these lines. Since $|\vec{u}| = |\vec{v}|$, the vector $\vec{u} + \vec{v} = (3,3)$ bisects the angle between $\vec{u}$ and $\vec{v}$. Equation of the required line is

$$\ell: \begin{cases} x = 3t+3 \\ y = 3t+4 \end{cases}.$$

QUESTION 2 (10 points)

Write an equation of the line of intersection of planes $x + y + z = 0$ and $y = z$.

Normal vectors of the planes in question are $\vec{N}_1 = (1,1,1)$ and $\vec{N}_2 = (0,1,-1)$. A direction vector of the line of intersection is $\vec{u} = \vec{N}_1 \times \vec{N}_2 = (-2,1,1)$. The point $(0,0,0)$ is a common point of the given planes. Then an equation of the intersection line is $(x, y, z) = (-2t, t, t)$.

QUESTION 3 (7 points)

Write the equations of changing the coordinates of an appropriate transformation which eliminates the xy term of the equation

$$x^2 + 4xy - 2y^2 - 6 = 0.$$

Characteristic equation is $2m^2 + 3m - 2 = 0$, whose roots are $m_1 = -2$ and $m_2 = \frac{1}{2}$. If we let $m = m_1 = \frac{1}{2}$, we get $\theta = \operatorname{atan}\frac{1}{2}$. A rotation of measure θ around the origin eliminates the xy term. To obtain the related equations of the rotation we first compute $\sin\theta = \frac{1}{\sqrt{5}}$ and $\cos\theta = \frac{2}{\sqrt{5}}$. Then

$$x = \frac{1}{\sqrt{5}}(2x' - y')$$
$$y = \frac{1}{\sqrt{5}}(x' + 2y').$$

QUESTION 4 (7 points)

Write the equations of changing the coordinates of an appropriate transformation which eliminates the terms x and y of the equation

$$16x^2 + 25y^2 - 64x + 50y - 115 = 0.$$

We can get rid of the linear terms by a translation. Assume that the system Oxy is translated to the point $O'(a, b)$ to define the coordinate system $O'XY$. Then $x = X + a$ and $y = Y + b$ which gives $16(X + a)^2 + 25(Y + b)^2 - 64(X + a) + 50(Y + b) - 115 = 0$ or

$$16X^2 + 25Y^2 + 32(a - 2)X + 50(b + 1) + F = 0.$$

By choosing $a = 2$ and $b = -1$, we can eliminate the linear terms. Equations of the translation are

$$x = X + 2$$
$$y = Y - 1.$$

<u>QUESTION 5</u> <u>(6 points)</u>

Find the canonical equation, eccentricity, focus and directrix of the conic

$$x^2 + 2x + y + 1 = 0.$$

The equation can be written as $(x+1)^2 + y = 0$. If we let $x = X - 1$ and $y = Y$ we obtain the equation $Y = -X^2$ of a parabola, hence eccentricity is 1. In $O'XY$ focus is at the point $\left(0, -\frac{1}{4}\right)$ and directrix is the straight line $Y = \frac{1}{4}$.

In Oxy focus is at $\left(-1, -\frac{1}{4}\right)$ and directrix is the line $y = \frac{1}{4}$.

<u>QUESTION 6</u> <u>(20 points)</u>

Let $\mathcal{E}$ and $\mathcal{H}$ be two central conics with eccentricities $1/\sqrt{2}$ and $\sqrt{2}$, respectively. If $\mathcal{E}$ and $\mathcal{H}$ are both centered at the origin and $F(4,0)$ is a common focus,

a) write an equation of $\mathcal{E}$,

The conic is an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a = \frac{f}{e} = 4\sqrt{2}$ and $b^2 = a^2 - f^2 = 16$. Then, the canonical equation of $\mathcal{E}$ is

$$\frac{x^2}{32} + \frac{y^2}{16} = 1.$$

b) write an equation of $\mathcal{H}$,

The conic is an hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a = \frac{f}{e} = 4\sqrt{2}/2$ and $b^2 = a^2 + f^2 = 24$. Then, the canonical equation of $\mathcal{H}$ is

$$\frac{x^2}{8} - \frac{y^2}{24} = 1.$$

c) find the intersection point(s) of $\mathcal{E}$ and $\mathcal{H}$.

We can write the equations of $\mathcal{E}$ and $\mathcal{H}$, respectively as

$$x^2 + 2y^2 = 32$$
$$3x^2 - y^2 = 24$$

which gives $x^2 = \frac{80}{7}$ and $y^2 = \frac{72}{7}$. There are four intersection points:

$$\left(\pm\sqrt{\frac{80}{7}}, \pm\sqrt{\frac{72}{7}}\right).$$

QUESTION 7 (20 points)

In each of the following cases, name and sketch the object which is defined by the given equation in $\mathbb{R}^3$.

a) $y - 3 = 0$,

b) $y^2 - z^2 = 4$,

c) $3x^2 + 4z^2 = 0$,

d) $z^2 - 8z = 0$,

e) $x^2 - 2y^2 + 3z^2 - 4 = 0$.