M E T U
Department of Mathematics

Math 115	**Analytical Geometry**	**Midterm II**	*Fall 2014–2015* *Date: 2.12.2014*
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1	2	3	4	5	**5** Questions on **4** Pages Total **70** Points	*Time: 17.40* *100 minutes*	total

READ THE PROBLEMS CAREFULLY AND GIVE DETAILED WORK

1. (6 pts) Given three vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ in $\mathbb{R}^3$, with $\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c}=0$, prove that $\overrightarrow{a}\times\overrightarrow{b}=\overrightarrow{b}\times\overrightarrow{c}=\overrightarrow{c}\times\overrightarrow{a}$.

Solution: $\overrightarrow{a}\times(\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c})=0 \quad\Longrightarrow\quad \overrightarrow{a}\times\overrightarrow{b}+\overrightarrow{a}\times\overrightarrow{c}=0 \quad\Longrightarrow\quad \overrightarrow{a}\times\overrightarrow{b}=\overrightarrow{c}\times\overrightarrow{a}$.

$(\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c})\times\overrightarrow{b}=0 \quad\Longrightarrow\quad \overrightarrow{a}\times\overrightarrow{b}+\overrightarrow{c}\times\overrightarrow{b}=0 \quad\Longrightarrow\quad \overrightarrow{a}\times\overrightarrow{b}=\overrightarrow{b}\times\overrightarrow{c}$.

Solution 2: Observe that $\overrightarrow{a}+\overrightarrow{b}=-\overrightarrow{c}$ means that the given vectors enclose a triangle. The area of this triangle can be written as

$$\frac{1}{2}|\overrightarrow{a}\times\overrightarrow{b}|=\frac{1}{2}|\overrightarrow{b}\times\overrightarrow{c}|=\frac{1}{2}|\overrightarrow{c}\times\overrightarrow{a}|.$$

On the other hand the direction of vectors $\overrightarrow{a}\times\overrightarrow{b}$, $\overrightarrow{b}\times\overrightarrow{c}$, and $\overrightarrow{c}\times\overrightarrow{a}$ are also the same, (these vectors are perpendicular to the plane of the triangle). Then the equality holds.

2. (12 pts) a) Suppose that $\overrightarrow{u}$ and $\overrightarrow{v}$ are vectors in $\mathbb{R}^3$, such that $|\overrightarrow{u}|=2$, $|\overrightarrow{v}|=3$, and $\overrightarrow{u}\times\overrightarrow{v}=(3,-3,3)$. Evaluate $|\overrightarrow{u}\cdot\overrightarrow{v}|$.

Solution: $(\overrightarrow{u}\cdot\overrightarrow{v})^2=|u|^2|v|^2\cos^2\theta=|u|^2|v|^2(1-\sin^2\theta)=|u|^2|v|^2(1-\frac{|\overrightarrow{u}\times\overrightarrow{v}|^2}{|u|^2|v|^2})$.

Then we have

$$(\overrightarrow{u}\cdot\overrightarrow{v})^2=|u|^2|v|^2-|\overrightarrow{u}\times\overrightarrow{v}|^2=4\cdot 9-27=9.$$

Finally, $|\overrightarrow{u}\cdot\overrightarrow{v}|=3$.

b) Find $|\mathrm{proj}_{\overrightarrow{u}}\overrightarrow{v}|$ (the length of projection of $\overrightarrow{v}$ to $\overrightarrow{u}$).

Solution: $|\mathrm{proj}_{\overrightarrow{u}}\overrightarrow{v}|=|\overrightarrow{v}|\cos\theta=\frac{|\overrightarrow{u}\cdot\overrightarrow{v}|}{|\overrightarrow{u}|}=\frac{3}{2}$.

3. (21 pts) Plane $\mathcal{P}$ contains the line $\ell_1 : \ x-1 = \frac{y-4}{-2} = \frac{z-5}{-1}$ and is parallel to the line $\ell_2 : \ x = \frac{y}{2} = \frac{z}{-1}$. (A plane and line are parallel if they have no common points).

a) Find an equation of plane $\mathcal{P}$.

Solution: $\overrightarrow{L}_1 = (1, -2, -1)$ and $\overrightarrow{L}_2 = (1, 2, -1)$ are direction vectors for ℓ_1 and ℓ_2 respectively.

$$\overrightarrow{N} = \overrightarrow{L}_1 \times \overrightarrow{L}_2 = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{vmatrix} = (4, 0, 4) \text{ is a normal vector for the plane } \mathcal{P}.$$

$P_0(1, 4, 5)$ is a point on ℓ_1, and ℓ_1 lies on $\mathcal{P}$. So, P_0 is also a point on $\mathcal{P}$.

An equation of $\mathcal{P}$ is $4(x-1) + 0(y-4) + 4(z-5) = 0$, or equivalently $4x + 4z - 24 = 0$, or

$$x + z - 6 = 0.$$

b) Find the distance $|P_1\mathcal{P}|$ from the point $P_1(1, 2-1)$ to the plane $\mathcal{P}$.

Solution: $|P_1\mathcal{P}| = \frac{|1+(-1)-6|}{\sqrt{1+1}} = \frac{6}{\sqrt{2}} = 3\sqrt{2}$.

c) Find the distance $|\ell_2\mathcal{P}|$ from line ℓ_2 to the plane $\mathcal{P}$.

Solution: Since $1 = \frac{2}{2} = \frac{-1}{-1}$, $\quad P_1(1, 2, -1)$ is a point on ℓ_2.

Therefore, $|\ell_2\mathcal{P}| = |P_1\mathcal{P}| = 3\sqrt{2}$.

OR

Since ℓ_2 and $\mathcal{P}$ are parallel, $\quad |\ell_2\mathcal{P}| = |P\mathcal{P}|$ for any point P on ℓ_2.

4. (10 pts) Find the values of α and β so that the line passing through the points $P(3,6,8)$ and $Q(1,\beta,-10)$ intersects the plane $\alpha x + y - 3z = 10$ at the point $R(2,3,-1)$.

Solution: Since R is a point on the plane $\alpha x + y - 3z = 10$,

$$2\alpha + 3 - 3(-1) = 10 \quad \Rightarrow \quad \alpha = 2\,.$$

Since P, Q and R are on the same line, $\overrightarrow{PQ}$ is parallel to $\overrightarrow{PR}$, that is $\overrightarrow{PQ} = k\overrightarrow{PR}$, where k is a constant. Then,

$$(1-3, \beta-6, -10-8) = k(2-3, 3-6, -1-8) \quad \Rightarrow \quad k = 2 \quad \text{and} \quad \beta - 6 = -3k\,.$$

Thus,

$$\beta = 6 - 3(2) = 0\,.$$

5. (21 pts) Consider line $d:\ x+y+2=0$, and point $F(1,1)$.

a) Find an equation of a parabola, C, with directrix d and focus F.

Solution: $P(x,y)\in C \quad \Longleftrightarrow \quad |PF|=|Pd| \quad \Longleftrightarrow \quad \sqrt{(x-1)^2+(y-1)^2}=\frac{|x+y+2|}{\sqrt{2}} \quad \Longleftrightarrow$

$2((x-1)^2+(y-1)^2)=(x+y+2)^2 \quad \Longleftrightarrow \quad 2x^2-4x+2+y^2-4y+2=x^2+y^2+2xy+4x+4y+4$

$$\Longleftrightarrow \quad x^2+y^2-2xy-8x-8y=0$$

b) Find the vertex V of the parabola and its axis ℓ.

Solution: Axis ℓ is perpendicular to d and passes through $(1,1)$, thus, its equation is

$\ell:(y-1)=k(x-1), \quad \text{where} \quad k\cdot(-1)=-1 \quad \Longleftrightarrow y=x.$

$V=\frac{1}{2}(F+G)$ where $G=d\cap\ell$. System $y=x$, $x+y+2=0$, gives $x=y=-1$, and $G(-1,-1)$.

So, $V=\frac{1}{2}((1,1)+(-1,-1)=(0,0)$.

c) Check if point $P(3,-1)$ lies on the parabola C.

Solution: Let $x=3, y=-1$. Then $x^2+y^2-2xy-8x-8y=9+1+6-24+8=0$.

So, P lies on the parabola.

d) Sketch d, F, V, and ℓ on the coordinate plane and indicate the position of C with respect to them.