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METU MATH 116, Final Examination

Date & time. June 7, 2011, 13:30–15:30 (120 minutes).

Instructors. Finashin, Pamuk, Pierce, Solak.

Problem 1 (12 points). Is $x^3 + x + 1$ reducible over:

(a) $\mathbb{Q}$?

Solution. $f(x) = x^3 + x + 1$ has degree 3 $\implies$ f is reducible if and only if f has a root (it concerns parts (a),(b), and (c)).

$\frac{p}{q} \in \mathbb{Q}$ is a root $\implies p|1,\ q|1 \implies \frac{p}{q} = \pm 1$.

$f(1) = 3,\ f(-1) = -1 \implies f(x)$ has no roots $\implies f$ is irreducible

(b) $\mathbb{Z}_3$?

Solution. $f(1) = 0 \in \mathbb{Z}_3, \implies f$ has a root $\implies f$ is reducible (divisible by $(x-1)$).

(c) $\mathbb{Z}_5$?

Solution. $f(0) = 1,\ f(1) = 3,\ f(2) = 11,\ f(3) = 31,\ f(4) = 69.$

No zeros modulo 5 $\implies f$ is irreducible over $\mathbb{Z}_5$.

Problem 2 (8 points). Letting $f(x) = 3x^4 + 5x^3 + x^2 + 5x - 2$, write $f(x)$ as a product of irreducible polynomials over $\mathbb{Q}$.

Solution. $\frac{p}{q}$ is a root of $f \implies p|2,\ q|3 \implies p \in \{\pm 1, \pm 2\},\ q \in \{\pm 1, \pm 3\}$ $\implies \frac{p}{q} \in \{\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}\}$.

$f(1) = 3 + 5 + 1 + 5 - 2 \neq 0,$
$f(-1) = 3 - 5 + 1 - 5 - 2 \neq 0,$
$f(2) = 48 + 40 + 4 + 10 - 2 \neq 0,$
$f(-2) = 48 - 40 + 4 - 10 - 2 = 0.$

-2 is a root $\implies f(x)$ is divisible by $(x+2)$. Using division algorithm,
$f(x) = (x+2)(3x^3 - x^2 + 3x - 1)$
$\frac{1}{3}$ is a root of $3x^3 - x^2 + 3x - 1$
$3x^3 - x^2 + 3x - 1 = (3x-1)(x^2+1).$

Answer: $f(x) = (x+2)(3x-1)(x^2+1)$
$(x+2),\ (3x-1)$ have degree 1 $\implies$ irreducible
$x^2 + 1 > 0$ for $x \in \mathbb{Q} \implies$ no rational roots $\implies$ irreducible.

Problem 3 (5 points). Is every integral domain a field? Explain.

Problem 4 (15 points). Find a polynomial $f(x)$ of least positive degree with the given properties. (Your answer should show the coefficients of $f(x)$.)

(a) $f(x)$ is over $\mathbb{C}$, and $f(2\mathrm{i}) = 0 = f(1 + \mathrm{i})$.

(b) $f(x)$ is over $\mathbb{R}$, and $2\mathrm{i}$ and $1 + \mathrm{i}$ are zeros of it.

(c) $f(x)$ is over $\mathbb{Z}_2$, and 1 (that is, [1]) is a zero of multiplicity 4.

Answers.

(a) $f(x) =$

(b) $f(x) =$

(c) $f(x) =$

Problem 5 (7 points). Over a field K, suppose $f(x)$ is a polynomial with no zeros in K. Must $f(x)$ be irreducible over K? Explain.

Solution. No. For example, the polynomial $x^4 + 2x^2 + 1$ over $\mathbb{R}$ has no zeros in $\mathbb{R}$ but it is reducible over $\mathbb{R}$.

Problem 6 (13 points). Working over $\mathbb{Z}_3$, letting

$$f(x) = x^5 + x + 1, \qquad\qquad g(x) = x^2 + 1,$$

find $s(x)$ and $t(x)$ such that $f(x) \cdot s(x) + g(x) \cdot t(x) = 1$.

Solution. $f(x) = x^5 + x + 1 = g(x)(x^2 + 2x) + 2x + 1$
$g(x) = x^2 + 1 = (2x + 1)(2x + 2) + 2$

Then,

$$2 = g(x) - (2x + 1)(2x + 2)$$
$$2 = g(x) - [f(x) - g(x)(x^3 + 2x)](2x + 2)$$
$$2 = f(x)(x + 1) + g(x)[1 + (x^3 + 2x)(2x + 2)]$$
$$2 = f(x)(x + 1) + g(x)(2x^4 + 2x^3 + x^2 + x + 1)$$

Hence,
$$1 = f(x)(2x + 2) + g(x)(x^4 + x^3 + 2x^2 + 2x + 2)$$

So, $s(x) = 2x + 2$ and $t(x) = x^4 + x^3 + 2x^2 + 2x + 2$.

Problem 7 (20 points). Let R be the subring $\{x + y\mathrm{i} \colon x, y \in \mathbb{Z}\}$ of $\mathbb{C}$, and let I be the ideal $\{x + y\mathrm{i} \colon x, y \in 2\mathbb{Z}\}$ of R.

(a) How many additive cosets has I in R? List them clearly.

(b) Is the quotient R/I cyclic as an additive group? Explain.

(c) Show that the function ϕ from R to $\mathbb{Z}_2$ given by

$$\phi(x + y\mathrm{i}) = [x + y] \qquad (*)$$

is a ring homomorphism.

(d) Does the same formula $(*)$ define a ring homomorphism from R to $\mathbb{Z}_3$? Explain.