M E T U
Department of Mathematics

	Basic Algebraic Structures Final Examination	
Code : *Math 116* Acad. Year : *2014-2015* Semester : *Spring* Instructors : *M.Bhupal, S.Finashin, F.Özbudak, E.Solak* Date : *05.June.2015* Time : *09:30* Duration : *100 minutes*	Last Name : Name : Student No : Department : Signature : **5** Questions on **5** Pages Total **80** Points	

1	2	3	4		

READ THE PROBLEMS CAREFULLY AND GIVE DETAILED WORK

1. (16 pts) a) Let R be a ring and I_1, I_2 be two ideals of R. For each of the following sets prove or disprove with a counterexample that they are ideals of R:

- $I_1 + I_2 = \{a + b : a \in I_1, b \in I_2\}$,
- $I_1 \cup I_2$,
- $I_1 \cap I_2$.

b) Let R be a ring and I_1, I_2 be two ideals of R. Prove that the set

$$I_1 I_2 = \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n : a_i \in I_1, b_i \in I_2, \text{ and } n \geq 1 \text{ is an integer}\}$$

is an ideal of R.

2. (16 pts) a) Let $R = \mathbb{Z}_{18}$ and consider the principal ideals $I_1 = (6) = \{6x : x \in \mathbb{Z}_{18}\}$ and $I_2 = (9) = \{9x : x \in \mathbb{Z}_{18}\}$. Describe the ideal $I_1 I_2$ (as defined in Question 1b above). Is it a principal ideal? Why?

b) Describe the quotient ring $\mathbb{Z}_{18}/(6)$, where (6) is the principal ideal as defined in Question 2a above.

3. (16 pts) Let $f(x), g(x) \in \mathbb{Z}_5[x]$ be the polynomials given by

$$f(x) = x^2 + 2x + 2 \text{ and } g(x) = x^3 + 3x^2 + 1.$$

a) Find the polynomial $h(x) = GCD(f(x), g(x)) \in \mathbb{Z}_5[x]$, that is the greatest common divisor of $f(x)$ and $g(x)$.

b) Using Euclidean Algorithm, find polynomials $s(x), t(x) \in \mathbb{Z}_5[x]$ such that

$$h(x) = f(x)s(x) + g(x)t(x).$$

4. (16 pts) Let $f(x) = x^4 - 6x^3 + 10x^2 + 2x - 15 \in \mathbb{Q}[x]$. Note that the polynomial $f(x)$ is also in the polynomial rings $\mathbb{R}[x]$ and $\mathbb{C}[x]$. Note also that $2 + i \in \mathbb{C}$ is a root of $f(x)$.

a) Factorize $f(x)$ over $\mathbb{C}$ (that is in $\mathbb{C}[x]$).

b) Factorize $f(x)$ over $\mathbb{R}$ (that is in $\mathbb{R}[x]$).

c) Factorize $f(x)$ over $\mathbb{Q}$ (that is in $\mathbb{Q}[x]$).

5. (16 pts) Let $M_2(\mathbb{Z})$ denote the ring of 2×2 matrices over $\mathbb{Z}$. Let $M_2(\mathbb{Z}_2)$ denote the ring of 2×2 matrices over $\mathbb{Z}_2$. Consider the map

$$\theta : M_2(\mathbb{Z}) \to M_2(\mathbb{Z}_2)$$

by

$$\theta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} [a] & [b] \\ [c] & [d] \end{bmatrix}.$$

Prove that θ is a (ring) homomorphism and describe the kernel of θ.