M E T U
Mathematics Department

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Math 260 – Basic Linear Algebra Final Exam		
2001–2002 **Fall Semester** **SF, HTK, EK, MP** **Monday, 14.1.2002** **13:00–15:30** **2.5 hours**	Name : Last Name : Signature :	Student Number : Section :
	5 questions on 6 pages.	

Q.1 20	Q.2 20	Q.3 20	Q.4 20	Q.5 20		Total 100
					SHOW ALL YOUR WORK.	

Q. 1. The first quadrant $V = \{\,(x,y) \in \mathbb{R}^2 \;:\; x > 0,\; y > 0\,\}$ in the plane is a real 2-dimensional vector space under the addition $\oplus$ and scalar multiplication $\odot$ defined by $(x,y) \oplus (z,w) = (xz, yw)$ and $r \odot (x,y) = (x^r, y^r)$.

(a) What is the zero element of V and what is the negative of (x,y)? Justify your answer.

(b) Is $P = \{\,(x,y) \in V \;:\; y = 1\,\}$ a subspace of V? What about $Q = \{\,(x,y) \in V \;:\; xy = 1\,\}$? How about $R = \{\,(x,y) \in V \;:\; y = 2x\,\}$? Justify your answer.

(c) Show that $B = \{(1,e),(e,1)\}$ is a basis for V.

(d) Show that $T : V \to \mathbb{R}$ given by $T(x,y) = 5\ln x$ is a linear transformation, where the target space $\mathbb{R}$ is with standard addition, scalar multiplication, and basis. Then find the matrix of T in the given bases.

Q. 2. (a) Determine the values of a and b for which the system $AX = B$ given by

$$\begin{aligned} x - 2y + az - w &= 1 \\ -x + y - z + w &= -1 \\ (a+1)y - a^2 z + aw &= 0 \\ (b+1)y - abz - a^2 w &= b \end{aligned}$$

has **(i)** no solution, **(ii)** unique solution, and **(iii)** infinitely many solutions. Write all solutions in the second and third cases.

(b) Considering A as the matrix of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ and using part **(a)**, find a basis for the kernel of T and a basis for the image of T.

Q. 3. **(a)** Find the eigenvectors and the corresponding eigenspaces of

$$A = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

(b) Is A invertible? Why?

(c) Find a matrix C with the property $C^{-1} = C^t$ such that $D = C^{-1}AC$ is diagonal and find D.

Q. 4. Let $f(x_1, x_2) = x_1^2 + 4x_1x_2 - 2x_2^2 + 2x_1 + 4x_2 + 1$.

(a) Find an orthogonal coordinate-change matrix P that diagonalizes the quadratic part of f.

(b) Write down formulas that express the old coordinates x_1 and x_2 in terms of the new coordinates y_1 and y_2.

(c) Write the equation $f(x_1, x_2) = 0$ in the (y_1, y_2)-coordinates in standard form by completing the square.

(d) Describe the set of points in the plane satisfying $f(x_1, x_2) = 0$.

Q. 5. Each part is a separate question independent of the other parts.

(a) Give an equation for $\mathrm{span}\{(-1, 1, 0), (-1, 0, 1)\}$ in $\mathbb{R}^3$.

(b) If a linear transformation $T : V \to V$ satisfies $T^2 = 0$, show that $\mathrm{Im}\, T \subset \mathrm{Ker}\, T$.

(c) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ takes the square S with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(2, 2)$ to the rhombus R with vertices $(0, 0)$, $(3, 1)$, $(3, -1)$, and $(6, 0)$. What are the possible values of the determinant of the matrix of T? Explain.

(d) Express $\det(\mathrm{cof}\, A)$ and $\mathrm{cof}(\mathrm{cof}\, A)$ in terms of $\det A$ and A.

(e) A linear transformation $T : V \to V$ satisfies $\|T(x)\| = \|x\|$ for all $x \in V$. What are the possible values of the eigenvalues of T? Use real scalars.