M E T U
Department of Mathematics

Group	BASIC LINEAR ALGEBRA Midterm II		List No.
Code : *Math 260* Acad. Year : *2007-2008* Semester : *Fall* Coordinator : *S.F, S.O, A.S.* Date : *November 28 2007* Time : *17:40* Duration : *90 minutes*	Last Name : Name : Department : Signature :	Student No. : Section :	
	4 QUESTIONS ON 4 PAGES TOTAL 100 POINTS		

1	2	3	4		

Show your work ! Partial credits will not be given for correct answers if they are not justified.

Question 1 (12+6+6=24 points)

The product of two (3×3)-matrices A and B is known to be $AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, and the adjoint matrix Adj(A) has determinant 4.

a) Find the determinants

$\det(A^{-1}) =$

$\det A =$

$\det B =$

Solution:

$(\det A)(\det B) = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = 24$ and $\mathrm{Adj(A)} = (\det \mathrm{A})\mathrm{A}^{-1}$

Since A is 3×3-matrix, $\det(\mathrm{Adj(A)}) = (\det \mathrm{A})^3 \det(\mathrm{A}^{-1}) = \frac{(\det \mathrm{A})^3}{\det \mathrm{A}} = (\det \mathrm{A})^2 = 4$

There are two possibilities: either $\det A = 2$, $\det B = 12$, $\det A^{-1} = \frac{1}{2}$, or $\det A = -2$, $\det B = -12$, $\det A^{-1} = -\frac{1}{2}$.

b) Can we conclude that the row $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a linear combination of the rows of matrix B ? (Present complete and detailed arguments !)

Solution:

Yes. The rows of matrix B form a basis of $\mathbb{R}^3$, because $\det B \neq 0$ (a theorem). So, **any row** and in particular $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a linear combination of the rows of matrix B.

c) Can we conclude that matrices A and B are row-equivalent ? (Present complete and detailed arguments !)

Solution:

Yes. A and B are both row equivalent to the unit matrix, because $\det A \neq 0$, $\det B \neq 0$ (a theorem). Thus, A and B are row equivalent to each other.

Question 2 (10+10=20 points)

Consider the subspace $V \subset \mathbb{R}^{2\times 3}$ formed by (2×3)-matrices A such that $A\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.

a) Find a basis of V and determine the dimension of V.

Solution:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix} a+2b \\ d+2e \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad \begin{matrix} a+2b=0 \\ d+2e=0 \end{matrix}$$

Four free variables b, c, e, f, so $\dim V = 4$.

Fundamental solutions (basis of V): $\begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) Extend your basis of V to a basis of $\mathbb{R}^{2\times 3}$ by choosing additional vectors from the standard basis of $\mathbb{R}^{2\times 3}$.

Solution:

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

So, additional vectors-matrices (corresponding to 5th and 8th columns) are $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Question 3 (8+8+8+8=32 points)

Consider the vector space of polynomials, $\mathcal{P}_2 = \{ax^2 + bx + c | a, b, c \in \mathbb{R}\}$, and its subset V formed by those $p(x) \in \mathcal{P}_2$ which satisfy the condition $p(1) + p'(2) = 0$.

a) Show that V is a subspace of $\mathcal{P}_2$.

Solution

$$\begin{array}{l} p_1(1) + p_1'(2) = 0 \\ p_2(1) + p_2'(2) = 0 \end{array} \Rightarrow (p_1 + p_2)(1) + (p_1 + p_2)'(2) = 0, \text{ so } V \text{ is closed with respect to addition}$$

$p(1) + p'(2) = 0 \Rightarrow cp(1) + cp'(2) = c(p(1) + p'(2)) = 0$, so V is closed with respect to scalar multiplication. Therefore V is a subspace of $\mathcal{P}_2$.

b) Show that $\mathcal{B} = \{p_1, p_2\}$ and $\mathcal{C} = \{q_1, q_2\}$ are two bases of V, where $p_1 = x^2 - 2x - 1$, $p_2 = x^2 - x - 3$, $q_1 = x - 2$, $q_2 = x^2 - 3x + 1$. Solution

if $p(x) = ax^2 + bx + c$, then $p'(x) = 2ax + b$ and $p(1) + p'(2) = (a + b + c) + (4a + b) = 5a + 2b + c = 0$. The solution space V is two dimensional (one equation with three unknowns a, b, c). This equation is satisfied for p_1, p_2, q_1, q_2:

$$\begin{array}{l} 5 - 4 - 1 = 0 \\ 5 - 2 - 3 = 0 \\ 0 + 2 - 2 = 0 \\ 5 - 6 + 1 = 0 \end{array}, \text{ so } p_1, p_2, q_1, q_2 \in V.$$

p_1, p_2 are not proportional and thus linearly independent. Then they form a basis of V (because V is two-dimensional). Similarly, q_1, q_2 form a basis.

c) Determine the coordinates of p_1, p_2 with respect to basis $\mathcal{C}$.

Solution

$$x^2 - 2x - 1 = c_1(x - 2) + c_2(x^2 - 3x + 1) \Rightarrow c_2 = 1, c_1 = 1,$$

$$\text{coordinates } [p_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x^2 - x - 3 = c_1(x - 2) + c_2(x^2 - 3x + 1) \Rightarrow c_2 = 1, c_1 = 2,$$

$$\text{coordinates } [p_2]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

d) Find the both transition matrices: $P_{\mathcal{B}\to\mathcal{C}}$ (from $\mathcal{B}$ to $\mathcal{C}$) and $P_{\mathcal{C}\to\mathcal{B}}$ (from $\mathcal{C}$ to $\mathcal{B}$). Write down how the coordinate columns $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{C}}$ are related to $P_{\mathcal{B}\to\mathcal{C}}$ and $P_{\mathcal{C}\to\mathcal{B}}$.

Solution

$$P_{\mathcal{C}\to\mathcal{B}} = [p_1]_{\mathcal{C}}[p_2]_{\mathcal{C}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$P_{\mathcal{B}\to\mathcal{C}} = P_{\mathcal{C}\to\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

Question 4 (8+8+8=24 points)

a) Show that $(p|q) = p(1)q(1) + p(2)q(2) + p(3)q(3)$ is an inner product in the space of polynomials $\mathcal{P}_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$.

Solution:

Positivity: $(p|p) = (p(1))^2 + (p(2))^2 + (p(3))^2 \geq 0$. If $(p|p) = 0$, then $p(1) = p(2) = p(3) = 0$, but polynomial p cannot have 3 roots (because its degree ≤ 2), except the case $p = 0$ (constant zero polynomial).

Symmetry: $(p|q) = (q|p)$ (obvious).

Linearity:
$(c_1p_1 + c_2p_2|q) = (c_1p_1(1) + c_2p_2(1))q(1) + (c_1p_1(2) + c_2p_2(2))q(1) + (c_1p_1(3) + c_2p_2(3))q(1) = c_1(p_1|q) + c_2(p_2|q)$.

b) Determine the norms $||1||$ and $||x||$ with respect to this inner product.

Solution:

$||p|| = \sqrt{(p|p)} = \sqrt{(p(1))^2 + (p(2))^2 + (p(3))^2}$
$||1|| = \sqrt{1 + 1 + 1} = \sqrt{3}$
$||x|| = \sqrt{1 + 4 + 9} = \sqrt{14}$

c) Determine $\cos(\phi)$, for angle ϕ between the vectors-polynomials 1 and x, with respect to the above inner product.

Solution:

$(1|x) = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 6$

$\cos(\phi) = \frac{(1|x)}{||1|| \cdot ||x||} = \frac{6}{\sqrt{3}\sqrt{14}} = \frac{6}{\sqrt{42}}$