Midterm 2 May 7
Solutions

Problem 1. (14pts) (a) State Euler's formula for graphs on surfaces. Under which condition on the graph this formula is satisfied ?

Answer: *$v - e + r = \chi(F)$ for a graph with v vertices, e edges and r regions on a surface F. It is true provided all the regions are simply connected (or equivalently, homeomorphic to a disc).*

(b) Give a definition of simplicial complex.

Answer: *A simplicial complex is a set S of simplices such that*

(1) *for any simplex $\sigma \in S$ any face of σ is also a simplex from S;*
(2) *two simplices from S may intersect only along their common face.*

(c) What is a chain complex ?

Answer: *A chain complex, C, is a sequence of abelian groups C_i, $0 \le i \le n$, related by homomorphisms $\partial_i \colon C_i \to C_{i-1}$, called differentials or boundary maps*

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0$$

so that the products $\partial_{i-1} \circ \partial_i$ of consecutive differentials vanish.

(d) What are the homology groups of a chain complex ?

Answer: *$H_i(C) = Z_i/B_i$, where $Z_i = \ker \partial_i$ is group of cycles and $B_i = \Im(\partial_{i+1})$ is group of boundaries.*

(e) How the boundary map of a simplicial chain complex is defined ?

Answer: *If $\sigma \in S$ is an k-dimensional simplex in a simplicial complex and $[\sigma]$ the corresponding generator of the chain group C_k of S, then*

$$\partial[\sigma] = \sum_{i=0}^{k} (-1)^i [\sigma_i],$$

where σ_i is the facet of σ obtaining by dropping its i-th vertex.

(f) What are the homology groups of surfaces F_g and N_k ?

Answer: *$H_0(F_g) = H_0(N_k) = \mathbb{Z}$ (as for any path-connected space)*
$H_1(F_g) = \mathbb{Z}^{2g}$, $H_1(N_k) = \mathbb{Z}_2 \oplus \mathbb{Z}^{k-1}$, $H_2(F_g) = \mathbb{Z}$, $H_2(N_k) = 0$.

(g) Which Δ-triangulation of a surface is a simplicial triangulation ?

Answer: *A Δ-triangulation of a surface is a simplicial if there is no*

(1) *loop-edges (i.e., the endpoints of an edge cannot coincide),*
(2) *multiple edges (i.e., there is no edges having the same endpoints),*
(3) *triangles sharing with another triangle more than one common edge.*

Problem 2. (5pts) *For the link diagram sketched below*
(a) find the topological type of its span,

(b) sketch a graph being a deformation retract of this span.

Answer:

Problem 3. (10pts) *Consider surface F glued from a hexagon according to the word $abacb^{-1}c^{-1}$. Determine if the following curves on the hexagonal model are one-sided or two-sided.*

(a) Line segment connecting the midpoints of the sides "a".

Answer:

(b) Line segment connecting the midpoints of the sides "b".

Answer:

(c) The diagonal separating sides ab from $acb^{-1}c^{-1}$.

Answer:

(d) The side a.

Answer:

(e) The side b.

Problem 4. (4pts) *Consider a graph with vertices A, B, C and edges $[AB]$, $[BC]$¡ and $[AC]$ as a simplicial complex, C.*
(a) For $x = 2[AB] - 3[BC] + [AC]$ find $\partial_1 x$.

(b) Give examples of a cycle in $C_1(C)$ and a boundary in $C_0(C)$.

Problem 5. (12pts) *Consider surface F obtained from a hexagon $ABCDEF$ by gluing side AB to DE and BC to FE. Divide the hexagon into triangles by diagonals AC, CE, and CF.*

(a) Is it a simplicial triangulation, or Δ-triangulation of F ? (Explain.)

Answer:

(b) What are the generators of the chain groups C_0, C_1 and C_2 ?

(c) Find the boundary of the chain $2[ABC] - [CDE]$.

(d) Calculate the homology group $H_2(F)$ using this chain complex.

Problem 6. (15pts) *Consider a polygonal cell complex X, whose 2-cells are represented by words $abcb^{-1}$, $acda^{-1}$, bcb.*

(a) Describe its chain groups C_i, $i = 0, 1, 2$, and the boundary maps $\partial_2 \colon C_2 \to C_1$ and $\partial_1 \colon C_1 \to C_0$.

(b) Find the homology groups $H_i(X)$, $i = 0, 1, 2$.