HOMEWORK I. EXERCISES TO THE TOPIC DIFFERENTIAL STRUCTURES

Problem 1. *Show that the following examples present smooth manifolds, determine their dimension and describe local coordinates in the charts of an atlas.*

(1) *a sphere in* $\mathbb{R}^3$,

(2) *a set*

$$\begin{aligned} 3x + y - z + u^2 &= a \\ x + y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

(for which values a it is a manifold ?)

(3) $SL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ *and* $SL(n, \mathbb{C}) \subset \mathbb{R}^{2n^2}$,

(4) $O_n \subset \mathbb{R}^{n^2}$ *and* $U_n \subset \mathbb{R}^{2n^2}$ *(Hint: determine the rank of the map* $f\colon A \to AA^*$*, first at the point* E*),*

(5) *Stiefel manifold* $S_{n,k}$ *consisting of orthonormal* k*-frames in* $\mathbb{R}^n$,

(6) *subset in* $\mathbb{C}^2$ *defined by the equations*

$$\begin{aligned} z_1^2 + z_2^2 &= a \\ |z_1|^2 + |z_2|^2 &= 1 \end{aligned}$$

(for which a is a manifold ?)

Problem 2. *Describe the charts and the transition functions in*

(1) $\mathbb{R}\mathrm{P}^n$,

(2) $S^2 \times S^1$

Problem 3. *Determine the dimensions of the Stiefel and the Grassmann manifolds* $S_{n,k}$ *and* $G_{n,k}$.

Problem 4. *Write explicitly the coordinate change formulae for the tangent and the cotangent vector bundles in* $\mathbb{R}\mathrm{P}^2$.

Problem 5. *Prove that* $f\colon X \to Y$ *is smooth iff it transforms smooth functions into smooth, that is* f^* *maps* $C^\infty(Y)$ *into* $C^\infty(X)$.

Problem 6. *Smooth functions in* $\mathbb{R}\mathrm{P}^2$ *are even smooth functions in* S^2 *(justify !). Prove that there exists a smooth embedding* $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4$*, or in the other words, there exist four smooth functions* $f_1, \dots, f_4$ *such that* $\forall x, y \in \mathbb{R}\mathrm{P}^2 \exists i : f_i(x) \neq f_i(y)$.