Middle East Technical University Differential Topology Math 541 Fall 2004

Final

(take-home)

Degrees of maps, transversal intersections and applications

Problem 1. *Construct a homotopically non-trivial map* $f\colon X \to X$ *of degree* 0 *if* X *is*

(1) *a closed oriented surface* F_g *of genus* $g > 0$*;*
(2) $S^n \times S^n$.
(3) *Explain why such an example does not exist for* $X = S^n$.
(4) *(advanced) Are there such examples for* $X = \mathbb{C}\mathrm{P}^2$ *?*

Problem 2. *Consider a smooth map,* $f\colon X \to Y_1 \times Y_2$, $f(x) = (f_1(x), f_2(x))$. *Let* $y_i \in Y_i$ *be regular values for* f_i *and* $M_i = f_i^{-1}(y_i)$, $i = 1, 2$. *Assuming that* $\dim X = \dim(Y_1 \times Y_2)$ *prove that* $\deg(f) = M_1 \circ M_2$.

Problem 3. *(1) Given a smooth map* $f\colon X^{n+m} \to S^n$ *and a regular value* $s \in S^n$, *prove that* $M = f^{-1}(s)$ *has a trivial normal bundle.*

(2) Prove the converse: if $M \subset X$ *is a codimension* n *submanifold having a trivial normal bundle, then there exists a smooth map* $f\colon X \to S^n$ *such that* $M = f^{-1}(s)$ *for some regular value* s.

(3) Prove that for any map $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^1$ *the pull-back* $f^{-1}(x)$ *of a regular value* $x \in \mathbb{C}\mathrm{P}^1$ *is a null-homologous submanifold.*

Problem 4. *Consider a closed connected surface* $F \subset S^3$ *and assume that a vector field* V *in* S^3 *has no singular points and is transverse to* F. *Show that* F *is homeomorphic to a torus.*

Problem 5. *Consider polynomials* $f\colon \mathbb{C}^p \to \mathbb{C}$ *and* $g\colon \mathbb{C}^q \to \mathbb{C}$ *having isolated singularities at* 0. *Define* $h\colon \mathbb{C}^{p+q} \to \mathbb{C}$ *as* $h(x, y) = f(x) + g(y)$. *Prove that* h *has also an isolated singularity at* 0 *and the Milnor number is additive, that is* $\mu_h(0) = \mu_f(0) + \mu_g(0)$.

Problem 6. *Prove that the following maps* f *have degree* 0.

(1) $f\colon S^3 \to S^1 \times S^2$ *and more generally* $f\colon S^{p+q} \to S^p \times S^q$, $p, q > 0$*;*
(2) $f\colon S^n \times S^m \to S^p \times S^q$, *if* $p, q, n, m > 0$, $p + q = n + m$, $\{p, q\} \neq \{n, m\}$*;*
(3) $f\colon X \to Y$ *which can be factorized through a sphere,* $X \to S^n \to Y$, *of dimension* $n \neq \dim X = \dim Y$*;*
(4) $f\colon X \to T^n$, *where* X *is a simply connected* n*-manifold and* T^n *is* n*-torus.*

Problem 7. *Assume that a map between oriented surfaces,* $f\colon F_1 \to F_2$ *has non-zero degree. Prove that* $g(F_1) \geqslant g(F_2)$.

Problem 8. *Prove that any map* $f\colon \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ *has even degree.*

(Problems 6(3–4) 7 and 8 require some knowledge of homology)