MATH 541 LECTURE I

Atlases and differential structures on a manifold

A manifold of dimension n is a topological space, M, which looks locally like an euclidian space $\mathbb{R}^n$, that is, $\forall x \in M$ there is a neighborhood of x, $U \subset M$, homeomorphic to $\mathbb{R}^n$. The homeomorphism $\phi\colon U \to \mathbb{R}^n$ is called *a chart* or *a coordinate system* around x. To avoid too exotic examples, it is required also that a manifold 1) Hausdorff, 2) has a countable basis of topology.

Exercises.

(1) *Construct a non-Hausdorff locally euclidian space.*
(2) *Give an example of locally euclidian space with a non-countable basis.*
(3) *The same as in (2), but the space should be connected.*

A *differential structure* in M allows to speak about differentiability of functions on M. Such a structure can be introduced if we fix *an atlas of charts* which agree with each other.

Recall that the class C^k is formed by those functions which are k times continuously differentiable. The class C^∞ is formed by infinitely many times differentiable functions, the latter functions will be called *smooth*. C^a means analyticity, C^0 means just continuity.

Two charts, $\phi_i\colon U_i \to \mathbb{R}^n$, are said to be C^k-compatible if the coordinates in these charts are related as C^k-functions in the common part of the domains. That is to say that the coordinate change map $\phi_{1,2} = \phi_2 \circ \phi_1^{-1}$ is a diffeomorphism of class C^k between the domains $V_i = \phi_i(U_1 \cap U_2) \subset \mathbb{R}^n$ The charts are automatically C^k-compatible if their domains U_i are disjoint.

A set of charts, $\mathcal{A} = \{\phi_i\colon U_i \to \mathbb{R}^n\}_{i\in I}$, is called *an atlas of class C^k*, if the domains U_i cover M, and the charts of $\mathcal{A}$ are pairwise C^k-compatible.

Remarks.

(1) An atlas of class C^k belongs also to all rougher classes, C^m, $m < k$ (m, k are from the ordered set $0, 1, 2, \dots, \infty, a$).
(2) Using a similar approach with complex charts $\phi\colon U \to \mathbb{C}^n$, one can define complex manifolds. However, for complex functions differentiable=analytic (holomorphic), so there is no notion of class C^k. Complex manifolds can be only analytic !

Examples.

(1) $\mathbb{R}^n$ has an atlas of class C^a with a unique chart, defined by the identity map $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$.
(2) sphere S^n has an atlas of class C^a consisting of a pair of charts, each of which is a stereographical progection, $S^n \smallsetminus \{x\} \to \mathbb{R}^n$.

(3) Projective space $\mathbb{RP}^n$ has a C^a-atlas with n charts, $\phi_i\colon U_i \to \mathbb{R}^n$, $\phi_i([x_0 : \cdots : x_n]) = (\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i})$, where $U_i \subset \mathbb{RP}^n$ is the complement of the hyperplane $x_i = 0$ ($\frac{x_i}{x_i}$ is omitted in the right hand side).

Assume that $\mathcal{A}$ is a C^k-atlas fixed for a manifold M. A function $f\colon M \to \mathbb{R}$ is called differentiable of class C^k (with respect to $\mathcal{A}$) if its representation, $f \circ \phi_i^{-1}\colon \mathbb{R}^n \to \mathbb{R}$ in any chart, $\phi_i\colon U_i \to \mathbb{R}^n$, of $\mathcal{A}$ is differential of class C^k. Similarly a map $f\colon M_1 \to M_2$ between two differential manifolds is differentiable of class C^k, if in any charts it is defined by differentiable functions. More precisely, for any charts $\phi\colon U \to \mathbb{R}^n$ of $\mathcal{A}_1$ and $\psi\colon V \to \mathbb{R}^m$ of $\mathcal{A}_2$ the composition $\psi \circ f \circ \phi^{-1}$ is differentiable of class C^k in the domain where this composition is defined (that is in $\phi(U \cap f^{-1}(V))$).

The above map f is called a diffeomorphism of class C^k if it is a homeomorphism and f together with f^{-1} belong to the class C^k.

Two C^k-atlases $\mathcal{A}_1$, $\mathcal{A}_2$ are said to be C^k-equivalent if their charts are pairwise C^k-compatible. Note that this is the same as to say that the union $\mathcal{A}_1 \cup \mathcal{A}_2$ is again a C^k-atlas.

Exercises.

(1) *Check that the above relation is really equivalence.*
(2) *Show that every C^k-equivalence class or atlases contains a unique maximal atlas (the latter is called sometimes the differential structure on a manifold M.*
(3) *Show that a pair of atlases $\mathcal{A}_i$, $i = 1, 2$, in M are C^k-equivalent if and only if the identity map is a C^k-diffeomorphism $(M, \mathcal{A}_1) \to (M, \mathcal{A}_2)$.*
(4) *Show that for the above equivalent C^k-atlases, the sets, $\mathbb{C}^k(M, \mathcal{A}_i)$, of differentiable functions coincide. Show the converse.*

Raising of the differentiability class of a manifold

Theorem. *For any C^k-atlas $\mathcal{A}$, $0 < n < \infty$, on a manifold M there exists a C^∞-atlas $\mathcal{A}'$ on M, which is C^k-compatible with $\mathcal{A}$.*

If $\mathcal{A}''$ is any other such a C^∞-atlas on M, then it defines essentially the same (C^∞-diffeomorphic) differential structure on M.

This result shows that there is no specific difference in the topology of manifolds of different classes of differentiability C^k, $1 \leqslant n \leqslant \infty$ (existence of C^1-atlas implies existence and essentially uniqueness of C^∞-atlas). That is why we will be dealing in what follows only with smooth (C^∞) manifolds.

Remark. On the contrary, topological manifolds are in general not smoothable, and smooth manifolds cannot be made analytic. So, we will exclude classes C^0 and C^a from the further considerations.

Different but diffeomorphic differential structures

Example. Any homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ gives a chart and thus defines a certain smooth structure in $\mathbb{R}$. It differs from the standard one, unless f is a diffeomorphism.

For example, consider an atlas $\mathcal{A}_1 = \{x^3\}$ formed by a uniques chart $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$. It is not compatible with the standard chart in $\mathbb{R}$, the identity map $x \mapsto x$, and thus $\mathcal{A}_1$ is not compatible with the standard atlas $\mathcal{A}_0 = \{x\}$.

Exercise. *(1) Show that the algebra of smooth functions $C^k(\mathbb{R}, \mathcal{A}_1)$ with respect to the atlas $\mathcal{A}_1$ is a subalgebra of the usual algebra $C^k(\mathbb{R}, \mathcal{A}_0)$ of smooth functions on $\mathbb{R}$.*

(2) Show that line $\mathbb{R}$ with the differential structures defined by $\mathcal{A}_1$ is diffeomorphic to the standard $\mathbb{R}$.

Remark. We conclude that $\mathcal{A}_0$ and $\mathcal{A}_1$ define distinct differential structures on $\mathbb{R}$, which are however equivalent up to a diffeomorphism. Such a distinction is not very crucial from the point of view of differential topology. It is much more essential to understand if the differential structures on a manifold are equivalent up to diffeomorphis.

For example, it turns out that any manifold of dimension $\leqslant 3$ has essentially (that is up to a diffeomorphism) unique differential structure. Beginning from the dimension 4 it is no longer true. On one hand, manifolds may have no smooth structure at all. On the other hand, there may be many such structures, essentially non-equivalent.

This phenomenon was discovered by J.Milnor in 1950s, who constructed famous 27 "exotic" 7-sphere (that is essentially non-equivalent differential structures on S^7).

Implicit presentation of differential manifolds

A subset $L \subset M$ in a smooth manifold M of dimension m is called a (smooth) l-dimensional submanifold of M if for any point $x \in M$ there exists a chart $\phi \colon U \to \mathbb{R}^m$ around x, such that $\phi(U \cap L) = \mathbb{R}^l$. This means that l looks in this coodinate system like a subspace $\mathbb{R}^l$ of $\mathbb{R}^m$.

Note that the restriction of ϕ is a chart on L mapping $V = U \cap L$ to $\mathbb{R}^l$.

Exercise. *Check that these charts agree. Define a differential structure on L. Show that the inclusion $L \to M$ is a smooth map.*

According to the **Whitney embedding theorem**, any smooth manifold can be considered lying in $\mathbb{R}^N$ (for sufficiently big N), that is diffeomorphic to a submanifold of $\mathbb{R}^N$.

Submanifolds in a manifold M most often appear as the locus $L = f^{-1}(y)$, where $y \in N$ is a regular value of some smooth mapping $f \colon M \to N$ $m = \dim M \geqslant n = \dim N$. We recall that $x \in M$ is called *a regular point* if the rank of the differential $d_x f$ has maximal rank, n, and *critical points* otherwise. If $f^{-1}(y)$ contains a critical point, then $y \in N$ is called *critical value.* Otherwise (if $f^{-1}(y)$ consists of regular values or empty) y is called *a regular value* .

Example. Consider $f \colon \mathbb{R}^m \to \mathbb{R}^n$, $x = (x_1, \dots, x_m)$ $f = (f_1, \dots, f_n)$, and $L = f^{-1}(0)$. At a regular point the Jacobi matrix $(\partial f_i / \partial x_j)$ has maximal rank n. By the implicit function theorem, there exists a neighborhood $U \subset \mathbb{R}^m$ (say, a ball) around x such that $U \cap L$ has 1-1 projection to one of the coordinate $(m-n)$-planes in $\mathbb{R}^m$, and the remaining n coordinates express as smooth functions of the chosen $(m-n)$ coordinates.

The projection to this $(m-n)$ plane provides a chart $L \cap U \to \mathbb{R}^{m-n}$.

Exercise. *Prove that such charts are compatible. Prove that L is a smooth manifold in case of arbitrary M.*

Exercise. *Prove that the smooth structure on* $S^n = \{x_0^2 + \cdots + x_n^2 = 0\} \subset \mathbb{R}^{n+1}$ *defined implicitly coincides with the structure defined earlier by stereographic projection.*

Homework. *In the following examples determine if the set is a smooth manifold. Find its dimension, the local coordinate systems.*

(1)
$$\begin{cases} 3x + y - z + u^2 &= a \\ x + y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{cases}$$

(2) *The Lie groups* $SL(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$ $SL(n,\mathbb{C}) \subset \mathbb{C}^{n\times n}$

(3) *Lie groups* $O(n)$ *and* $U(n)$.

(4) *Stiefel manifold* $S_{n,k}$ *formed by all* k*-frames in* $\mathbb{R}^n$.