Final, Part 1

June 6, 32 points in 4 questions

Name and the student number:

Q1. (10 pts)

(1) *What does mean "map $f\colon M \to N$ is transversal to a submanifold $L \subset N$" ?*

(2) *State the theorem about the preimage of a submanifold with respect to a map transversal to that submanifold.*

(3) *What is the exponential map ?*

(4) *What is a tubular neighborhood of a submanifold ?*

(5) *What is a smooth homotopy ?*

(6) *What is a Riemannian metric in a vector bundle E ? How it can be defined as a tensor field ?*

(7) *State the Whitney embedding theorem.*

(8) *What is a cobordism from manifold M_0 to manifold M_1 ? In which case it is called "oriented" ?*

Q2. (8 pts) *Give a more simple description of manifolds obtained by the following constructions.*

(a) The double of $S^2 \times S^3$.

(b) The connected sum $\mathbb{C}\mathrm{P}^2 \# S^4$.

(c) The connected sum of two tori T^2.

(d) The double of the Möbius band.

(d) A tubular neighborhood of S^1 *in* $\mathbb{R}^2$.

(e) A tubular neighborhood of a knot (S^1 *embedded in* $\mathbb{R}^3$*).*

Q3. (8 pts) *Explain why any closed surface embedded into* $\mathbb{R}^3$ *is orientable.*

Q2. (8 pts) *(a) Map* f *from* 15*-dimensional manifold* M *to* $\mathbb{R}^{20}$ *is transverse to a* 13*-dimensional submanifold* $L \subset \mathbb{R}^{20}$. *What is the dimension of* $f^{-1}(L)$ *?*

(b) For a given homotopy $X \times [0,1] \to \mathbb{C}\mathrm{P}^2 \times S^5$, *the preimage* $f^{-1}(y)$ *of a regular value* y *is* $\mathbb{R}\mathrm{P}^2$. *Find the dimension of* X.

(c) In the previous question, show that X *cannot be orientable.*

Q3. (8 pts) *(a) Show that a sphere* S^{n-1} *in* $\mathbb{R}^n$ *has a trivial normal bundle.*

(b) Knowing that S^3 *is parallelizable (which means that its tangent bundle* TS^3 *is trivial) show that its self-intersection in* TS^3 *is zero.*

(c) Show that for any parallelizable manifold X *its Euler characteristic* $\chi(X)$ *vanishes.*

(d) Knowing that $\chi(X) = 0$ *deduce that the self-intersection index of the diagonal* $\Delta_X \subset X \times X$ *is zero.*

(e) Find the self-intersection index of the diagonal $\Delta_F \subset F \times F$ *where* F *is a closed oriented surface of genus* 2.

Q5. (10 pts) *Knowing that a vector field* V *in* $\mathbb{C}\mathrm{P}^2$ *has only non-degenerate zeros, estimate from below the number of zeros.*

Q5. (10 pts) *(a) Map* $f\colon X \to Y$ *is null-homotopic and* $g\colon X \to Z$ *is any map,* $\dim X = \dim Y + \dim Z = n$. *Prove that the map* $X \to Y \times Z$, $x \mapsto (f(x), g(x))$ *has degree* 0.

(b) Show that any map $S^n \to S^1 \times X$, $\dim X = n-1$, *has degree* 0.