PROBLEMS IN HOMOLOGY THEORY

Problem 1. *Construct an example of a map which is not null-homotopic, but induces a trivial map in homology.*

Problem 2. *Prove that for any closed manifold M of dimension $n > 0$ there exists $i \leqslant n$ such that $\pi_i(M) \neq 0$.*

Problem 3. *Prove that $\mathbb{C}\mathrm{P}^3$ and $S^2 \times S^4$ are homotopy non-equivalent.*

Problem 4. *Prove that the multiplication in the ring $H^*(\Sigma X)$ is trivial for suspension ΣX over any X.*

Problem 5. *Prove that a group G cannot act on a contractible cell complex X if (a) X is a finite complex; (b) X is finitely-dimensional (possibly infinite) and G is a finite group.*

Problem 6. *Prove that $[K, S^n] = H^n(K, \mathbb{Z})$, if $\dim K = n$.*

Problem 7. *(a) Prove that $\mathbb{C}\mathrm{P}^n \smallsetminus S^1 \cong \mathbb{C}\mathrm{P}^{n-1} \vee S^{2n-2}$. (b) Determine S^n/S^k and $S^n \smallsetminus S^k$ up to homotopy equivalence. (c) Prove that $X \wedge S^1 = \Sigma X$. Here $X \wedge S^1 = (X \times S^1)/(X \vee S^1)$.*

Problem 8. *Assume that the universal covering of M is $\mathbb{R}^n$. Prove that $[X, M] = \mathrm{Hom}(\pi_1(X) \to \pi_1(M))$ for any path-connected X.*

Problem 9. *(a) Determine the ring structure in $H^*(\mathbb{C}\mathrm{P}^2 \smallsetminus A_3)$ and $H^*(\mathbb{C}\mathrm{P}^2, A_3)$ for a non-singular cubic $A_3 \subset \mathbb{C}\mathrm{P}^2$. (b) Determine the structure of a module over $H^*(\mathbb{C}\mathrm{P}^2)$ in these rings. (c) Determine the ring structure in the cohomology of $\mathbb{R}\mathrm{P}^3/\mathbb{R}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^3/\mathbb{C}\mathrm{P}^1$.*

Problem 10. *Describe the transfer in the Gysin sequence for the natural fibrations (1) $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$, (2) $L(n,1) \to S^2$, (3) $\det\colon U(n) \to U(1)$, (4) $\mathbb{C}\mathrm{P}^3 \to \mathbb{H}P^1 = S^4$.*

Problem 11. *Determine the transfer map $f_!$ in homology for the following inclusion maps (a) $\mathbb{R}\mathrm{P}^3 \to \mathbb{C}\mathrm{P}^3$, (b) $\mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^3$, (c) $A_n \to \mathbb{C}\mathrm{P}^2$ (A_n is a non-singular curve of degree n), (d) $Q \to \mathbb{C}\mathrm{P}^3$ (Q is a non-singular quadric).*

Problem 12. *What does Lefschetz fixed point theory says about possible number of fixed points of maps $\mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$, $S^2 \times S^4 \to S^2 \times S^4$?*

Problem 13. *Consider a map $f\colon T^2 \to T^2$ with $f_*\colon \pi_1(T^2) \to \pi_1(T^2)$, $f_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Assuming that f has finitely many fixed points, determine their (algebraic) number.*

Problem 14. *Determine the obstruction for a section in the following bundles: (a) $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$, (b) $S^n \to \mathbb{R}\mathrm{P}^n$, (c) $LF_g \to F_g$, where LF_g is the space of linear elements on a genus g surface F_g.*

Problem 15. *What can you say about the homotopy classes of maps, $[S^2 \times S^2 \to S^2]$, $[S^2 \times S^2 \to S^3]$, $[\mathbb{C}\mathrm{P}^2 \to S^2]$, $[\mathbb{C}\mathrm{P}^3 \to S^2 \times S^4]$, using the obstruction theory ?*

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