PROBLEM LIST III

Pontryagin-Thom construction

Problem 1. *A regular fiber $p^{-1}(y)$ for a map $p\colon S^{n+2} \to S^n$, $n \geqslant 4$, is homeomorphic to S^2. Deduce from the Pontryagin theorem that p is null-homotopic.*

Problem 2. *Let $f\colon S^{n+3} \to S^n$, $n \geqslant 2$, and $M = f^{-1}(S^1)$, a regular pull-back of a standard circle. Show that M is not homeomorphic to (a) $\mathbb{R}\mathrm{P}^4$, (b) $\mathbb{C}\mathrm{P}^2$.*

Problem 3. *Given $f\colon S^{n+4} \to S^n$, $n \geqslant 1$, show that a regular fiber $f^{-1}(y)$, $y \in S^n$, cannot be (a) $\mathbb{R}\mathrm{P}^4$, (b) $\mathbb{C}\mathrm{P}^2$, (c) $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$.*

Problem 4. *Given $f\colon S^{n+1} \to S^n$, $S^{n+2} \to S^n$ and a degree 2 map $h\colon S^n \to S^n$, prove that $h \circ f$ and $h \circ h$ are null-homotopic for sufficiently big n. Specify the estimate on n.*

Problem 5. *Let $H_n\colon S^3 \to S^2$ denotes a map with the Hopf invariant n. Show that a CW-complex X obtained by gluing a pair of 3-cells to S^2 along H_2 and H_3, is homotopy equivalent to $\mathbb{C}\mathrm{P}^2 \wedge S^3$.*

Problem 6. *Let $f_n = \Sigma^{n-2} f_2\colon S^{n+1} \to S^n$ be the suspension over the Hopf map f_2 and $g_n = f_n \circ f_{n+1}\colon S^{n+1} \to S^n$, $h_n = f_n \circ f_{n+1} \circ f_{n+2}\colon S^{n+3} \to S^n$.*

(1) *Describe regular fibers of f_n, g_n and h_n.*
(2) *Show that g_n represent a generator of $\pi_{n+2}(S^n) = \mathbb{Z}/2$ for sufficiently big n (specify how big should be n).*

Obstruction Theory

Problem 7. *Determine an obstruction to extend a degree n map $f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ to a map $F\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^1$.*

Problem 8. *Let $f_n\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$, $[z_0 : z_1 : z_2] \mapsto [z_0^n : z_1^n : z_2^n]$.*

(1) *Determine $f_n^*\colon H^*(\mathbb{C}\mathrm{P}^2) \to H^*(\mathbb{C}\mathrm{P}^2)$.*
(2) *Describe the first obstruction $o(f_n, f_m)$.*
(3) *determine the Euler number of the bundle $f_n^*\xi|_{\mathbb{C}\mathrm{P}^1}$, induced from the canonical bundle ξ over $\mathbb{C}\mathrm{P}^2$.*
(4) *Describe the bundles $f_n^*(T)|_{\mathbb{C}\mathrm{P}^1}$ and $f_n^*(\det)|_{\mathbb{C}\mathrm{P}^1}$, where T and $\det$ are respectively the tangent and the determinant bundles on $\mathbb{C}\mathrm{P}^2$.*