PROBLEM LIST IV

Sheaves

Problem 1. *Let $\mathcal{O}_X$ denotes the sheaf of smooth functions on a smooth manifold X, that is $\mathcal{O}_X(U) = \{f \colon U \to \mathbb{R}\}$ (where f is smooth). Prove that $\mathcal{O}_X$ determine the differential structure on X. Specifically,*

(1) *describe a differential structure on X in terms of the sheaf $\mathcal{O}_X$;*
(2) *formulate the condition under which a map $h \colon X \to Y$ is a diffeomorphism in term of sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$ on smooth manifolds X and Y;*
(3) *find the conditions under which a subsheaf of the sheaf of continuous functions on a manifold X defines a differential structure on X.*

Problem 2. *Let $\mathcal{O}^*_X$ denotes the sheaf of $\mathbb{C}^*$-valued continuous functions on X (that is $\mathcal{O}^*_X(U) = \{f \colon U \to \mathbb{C}^*\}$). Check that the transition functions for a complex line bundle over X define a 1-cocycle in $\mathcal{O}^*_X$ (hint: find first a suitable open covering of X and define a cycle corresponding to it, then pass to the limit). Set up a correspondence between $H^1(X, \mathcal{O}^*_X)$ and the isomorphism classes of line bundles over X.*

Problem 3. *Construct the complex $C_{I,\mathcal{F}}$ for the standard covering of $\mathbb{R}\mathrm{P}^2$ by affine charts $\{U_0, U_1, U_2\}$ and the locally constant sheaf $\mathcal{F}$ with $\mathcal{F}_x = H_2(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^2 \smallsetminus \{x\})$. Determine the homology of this complex.*

Problem 4. *Let $\mathcal{O}_A$ denotes the sheaf of continuous functions on A. Describe explicitly the steaks of its direct image $i_*\mathcal{O}_A$ under the inclusion map $i \colon A \to X$. Consider in particular examples of inclusions $[0,1) \to \mathbb{R}$ and $[0,1] \to \mathbb{R}^2$.*

Problem 5. *Given a map $f \colon X \to Y$ and sheaves $\mathcal{F}_X$ and $\mathcal{F}_Y$ on X and Y respectively, construct natural homomorphisms*

$$H^*(Y, f_*\mathcal{F}_X) \to H^*(X, \mathcal{F}_X)$$
$$H^*(Y, \mathcal{F}_Y) \to H^*(X, f^*\mathcal{F}_Y)$$