Middle East Technical University Fall 2002 Math 744 Fundamental Techniques in Differential Topology

PROBLEM LIST VI (TAKE-HOME MIDTERM II)

Heegaard splitting and Kirby Calculus

Problem 1. *Determine how the group $H_1(M)$ of a 3-manifold M changes after surgery along a knot $K \subset M$ with framing n. In particular*

1. *consider the case of $M = S^3$;*
2. *show that if M is $\mathbb{Z}HS$ then ± 1-framed surgery along any knot K gives again $\mathbb{Z}HS$;*
3. *show that $H_1(M)$ does not change under ± 1-framed surgery (knot K here is supposed to be null-homologous in M and framing is taken with respect to some Seifert surface for K);*
4. *show that all Seifert surfaces for K in any 3-manifold M determine the same framing of K.*

Problem 2. *Knowing the critical points of Morse functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ determine the ones for $f \circ p_X + g \circ p_Y \colon X \times Y \to \mathbb{R}$, where p_X, p_Y are the projections. In particular*

1. *estimate the Heegaard genus of $S^1 \times F$, where F is an oriented closed surfaces of genus g;*
2. *describe explicitly a Heegaard splitting of $S^1 \times F$ of genus $2g+1$ (hint: note that $F^\circ \times [0,1]$ is a handlebody of genus $2g$, where F° is obtained from F by removing a small 2-disk).*
3. *describe a similar Heegaard splitting for a mapping torus M_f, where $f\colon F \to F$.*

Problem 3. *Show that the four framed links sketched below describe the same 3-manifold.*

Problem 4.

1. *If in a framed link $\mathcal{L}$ a component L_0 is an unknot with framing 0 which links only one other component L_1 geometrically once, then $L_0 \cup L_1$ may be moved away from the link $\mathcal{L}$ without changing framings.*
2. *Show that the two-component link $\mathcal{L}' = L_0 \cup L_1$ defines S^3, observing that a handle slide of L_1 along L_0 preserves the link changing only the framing of L_1 by ± 2.*
3. *Deduce from the above observations that $M_{\mathcal{L}} \cong M_{\mathcal{L} \smallsetminus \mathcal{L}'}$, that is, if we remove L_0 and L_1 from $\mathcal{L}$ than the 3-manifold described by the link will not change.*