Test 1

(Take-home)

Q1. (5 pts) *Let* $\omega = e_1^* \wedge e_2^* + 2e_2^* \wedge e_3^* - (e_1^* + e_3^*) \wedge e_4^* + 5e_3^* \wedge e_1^* \in \Lambda^2(\mathbb{R}^4)^*$, *where* $\{e_i\}$ *is a basis of* $\mathbb{R}^4$.

(1) *Find a matrix* (ω_{ij}) *of* ω, *as it is viewed as a bilinear form in the given basis of* $\mathbb{R}^4$.
(2) *Find the images of vectors* e_i *under the map* $\widetilde{\omega}$.
(3) *Determine if* ω *is non-degenerate.*
(4) *Find* $\omega \wedge \omega$.
(5) *Find the Pfaffian of the matrix* (ω_{ij}).

Q2. (3 pts) *Given a symplectic form* $\omega = e_1^* \wedge e_2^* + e_2^* \wedge e_3^* + e_3^* \wedge e_4^*$ *in* $\mathbb{R}^4$

(1) *Find* $\omega(v, w)$, *where* $v = (1, 1, 1, 2)$ *and* $w = (1, 3, 1, 3)$. *Is* $\operatorname{span}(v, w)$ *a Lagrangian plane ?*
(2) *Find a Lagrangian plane contained is the subspace* $\operatorname{span}(e_1, e_2, e_3)$ *(that is, defined by equation* $x_4 = 0$*).*
(3) *Find a Lagrangian plane containing vector* $e_1 + e_2 + e_3 + e_4$.

Q3. (3 pts) *Assume that* $b\colon V \times V \to \mathbb{R}$ *is a nondegenerate bilinear (for simplicity, symmetric or alternating) form in a vector space* V *of dimension* n, *and*

$$W^b = \{v \in V \mid b(v, w) = 0, \quad \forall w \in W\}.$$

(1) *Show that the restriction* $b|_W$ *is degenerate if and only if there exists* $v \in W$, $v \neq 0$, *such that* $b(v, w) = 0$ *for all* $w \in W$. *And the latter is equivalent to that* $W \cap W^b \neq \{0\}$.
(2) *Assuming that* $b|_W$ *is non-degenerate, show that for any* $v \in V$ *there exists* $w \in W$ *such that* $v - w \in W^b$.
(3) *Show using the results above that* V *is a direct sum of* W *and* W^b *if* $b|_W$ *is non-degenerate, and deduce that* $\dim W^b + \dim W = n$.

Q4. (2 pts) *Consider a subspace* $W \subset V$ *in a vector space* V *and define*

$$W^\perp = \{\alpha \in V^* \mid \alpha(v) = 0 \;\; \forall v \in W\} \subset V^*.$$

Prove that $W \times W^\perp$ *is a Lagrangian subspace of* $V \times V^*$ *with respect to the canonical symplectic form* ω_{can} *defined on* $V \times V^*$ *as follows:*

$$\omega_{can}((v, \alpha), (w, \beta)) = \beta(v) - \alpha(w)$$

Q5. (4 pts) *Prove that the intersection of any pair of the groups* $\mathrm{O}(\mathbb{R}^{2n})$, $\mathrm{GL}(\mathbb{C}^n)$, *and* $\mathrm{Sp}(\mathbb{R}^{2n})$ *is contained in the third group. Deduce that the intersection of any pair of these groups is the unitary group* $\mathrm{U}(\mathbb{C}^n)$.

Recall the definitions: for a linear operator $T\colon V \to V$

(1) $T \in \mathrm{O}(V)$ (T is orthogonal with respect to an inner product $(\langle *, * \rangle)$ in V), if

$$\langle Tv, Tw \rangle = \langle v, w \rangle \;\; \forall v, w \in V,$$

(2) $T \in \mathrm{GL}(V)$ (T is complex with respect to a complex structure $J\colon V \to V$), if

$$T(Jv) = J(Tv) \;\; \forall v \in V,$$

(3) $T \in \mathrm{Sp}(V)$ (T is symplectic with respect to a symplectic structure ω), if

$$\omega(Tv, Tw) = \omega(v, w) \;\; \forall v, w \in V.$$

Hint: Note also that the complex structure in $\mathbb{R}^{2n} \cong \mathbb{C}^n$, the dot product and the canonical symplectic form are related by the relation

$$\omega(v, w) = \langle v, Jw \rangle, \;\; \forall v, w \in \mathbb{R}^{2n}$$