

RELATIVE SEIBERG-WITTEN AND OZSVÁTH-SZABÓ INVARIANTS FOR SURFACES EMBEDDED IN 4-MANIFOLDS

SERGEY FINASHIN

§1. INTRODUCTION

1.1. The subject. In this paper we define and study a relative Seiberg-Witten (SW) and a similar relative Ozsváth-Szabó (OS) invariant for surfaces Σ of genus $g > 1$ embedded in a 4-manifold X with self-intersection index $\Sigma^2 \geq 0$. Recall that Taubes [T5] constructed such a relative invariant, $SW_{X,\Sigma}$, for $g = 1$, $\Sigma^2 = 0$ analyzing the solution space to the Seiberg-Witten equation on a compact manifold X° obtained after removing a tubular neighborhood of Σ from X . It turned out that his invariant $SW_{X,\Sigma}$ is basically the same as for the fiber sum, $SW_{X\#_\Sigma E(1)}$. We use a similar property as the definition of $SW_{X,\Sigma}$ in the case of $g > 1$, which makes things look more elementary, although the result is probably the same as might be following the traditional approach in the spirit of Taubes.

More precisely, we define $SW_{X,\Sigma}$ as the pull-back of $SW_{X\#_\Sigma W}$, where W admits a relatively minimal Lefschetz fibration $W \rightarrow S^2$ with a fiber Σ and with the Betti number $b_1(W) = 0$ (existence of such W is obvious). Under such assumptions, the choice of W turns out to be not essential, which is our first observation. The standard set of properties for the SW invariant is naturally inherited by its relative version, enjoying some improvements. For example, the product formula for $SW_{X,\Sigma}$ looks more simple, just as the product of polynomials, like in the case $g = 1$ of Taubes (recall that its simplicity in that case had as a consequence the remarkable results in [FS2]). Another improvement concerns the adjunction inequality, which becomes applicable to membranes (surfaces with the boundary on Σ), and does not require the positivity assumption for their self-intersection. The invariant $SW_{X,\Sigma}$ appears to be a refinement of SW_X restricted to the set $\text{Spin}^{\mathbb{C}}(X, \mathfrak{s}_\Sigma) = \{x \in \text{Spin}^{\mathbb{C}}(X) | c_1(\mathfrak{s}) = \chi(\Sigma) + \Sigma^2\}$, which is the image of the forgetful map $\text{abs}_{X,\Sigma}: \text{Spin}^{\mathbb{C}}(X, \Sigma) \rightarrow \text{Spin}^{\mathbb{C}}(X)$. Namely, if $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X, \mathfrak{s}_\Sigma)$, then $\text{abs}_{X,\Sigma}^{-1}(\mathfrak{s})$ contains several relative basic structures, $\mathfrak{r}_i \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$, and $SW_X(\mathfrak{s}) = \sum_i SW_{X,\Sigma}(\mathfrak{r}_i)$. The invariant $SW_{X,\Sigma}$ can be easily calculated in many interesting examples, thanks to the paper [FS2] containing some nice computation of $SW_{X\#_\Sigma W}$.

The same strategy applied to the OS invariant of closed 4-manifolds yields a similar relative invariant $OS_{X,\Sigma}$. Note that the latter yields a refinement not only of the usual OS invariant for X , but also for the complement $X^\circ = X \setminus N(\Sigma)$ of a tubular neighborhood $N(\Sigma)$ of Σ . One can also construct in a similar spirit a refinement of the OS invariant for some 4-manifolds whose boundary differs from $\Sigma \times S^1$, as it is sketched in 5.3 (relativization with respect to a surface $\Sigma \subset X$ is replaced there by relativization with respect to a boundary component of X). A generalization of this constructions to all 4-manifolds with boundary is likely to give a true TQFT for $\text{Spin}^{\mathbb{C}}$ manifolds.

The manifolds that we consider in what follows are smooth, oriented, connected, and closed, unless we state otherwise. We suppose also that the surface $\Sigma \subset X$ has genus $g > 1$

and either essential ($[\Sigma] \in H_2(X)$ has infinite order) with self-intersection $\Sigma^2 = 0$, or has $\Sigma^2 > 0$ (in the latter case we blow it up to obtain $\Sigma^2 = 0$). This implies in particular that $b_2^+(X) \geq 1$.

1.2. The Seiberg-Witten and the Ozsváth-Szabó invariants. In its simplest version, the SW invariant of a 4-manifold X is a function on the set of $\text{Spin}^{\mathbb{C}}$ structures, $SW_X: \text{Spin}^{\mathbb{C}}(X) \rightarrow \mathbb{Z}$, which takes non-zero values only at a finite set of $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X)$, called *the basic structures*, whose degree $d(\mathfrak{s}) = \frac{1}{4}(c_1^2(\mathfrak{s}) - (2\chi(X) + 3\sigma(X)))$ is zero. The corresponding Ozsváth-Szabó invariant (extracted as the reduced form of $\Phi_{X,\mathfrak{s}}$ in [OS4], §4) is an analogous function $OS_X: \text{Spin}^{\mathbb{C}}(X) \rightarrow \mathbb{Z}/\pm$ well-defined up to sign. The sign of $SW_X(\mathfrak{s})$ depends on the choice of an orientation of $H^1(X; \mathbb{R}) \oplus H_+^2(X; \mathbb{R})$ called the *homology orientation*, which is supposed to be fixed. In a bit special case of $b_2^+(X) = 1$ the invariants SW_X , OS_X depend on some additional data that must be fixed: SW_X depends on an orientation of the line $H_+^2(X)$ (see [KM], [MST], or [T1]), while OS_X , according to [OS4, Prop. 2.6], depend on the choice of an isotropic line in $H_2(X)$.

One can consider also a refined version of SW_X and OS_X , which may take non-zero values for $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X)$ of degree $d(\mathfrak{s}) \geq 0$. These values are homomorphisms $SW_{X,\mathfrak{s}}: \mathbb{A}_X \rightarrow \mathbb{Z}$, $OS_{X,\mathfrak{s}}: \mathbb{A}_X \rightarrow \mathbb{Z}/\pm$, supported in the set of homogeneous elements of degree $d(\mathfrak{s})$ from the graded ring $\mathbb{A}_X = \Lambda(H_1(X)/\text{Tors}) \otimes \mathbb{Z}[U]$, where Λ stands for the exterior algebra and the grading is defined on the generators, so that U has degree 2 and $\alpha \in H^1(X) \setminus \{0\}$ have degree 1. The dual consideration, which is more convenient for us, interprets the refined version of SW_X as a map $\text{Spin}^{\mathbb{C}}(X) \rightarrow \mathbb{A}_X^* = \Lambda(H^1(X)) \otimes \mathbb{Z}[U] \cong \Lambda(H^1(X; \mathbb{Z}[U]))$, such that $SW_X(\mathfrak{s})$ is homogeneous of degree $d(\mathfrak{s})$. Reducing the values of OS_X modulo 2, we obtain a similar map $\text{Spin}^{\mathbb{C}}(X) \rightarrow \mathbb{A}_X^* \otimes \mathbb{Z}/2$.

We will use notation $S_X: \text{Spin}^{\mathbb{C}}(X) \rightarrow R_X$ for any of the invariants SW_X or OS_X , either in the refined or in the reduced form. R_X here is a ring \mathbb{A}_X^* or $\mathbb{A}_X^* \otimes \mathbb{Z}/2$ in case of the refined forms of SW_X or OS_X . In case of the reduced forms, R_X is just \mathbb{Z} or $\mathbb{Z}/2$. The relative version, $R_{X,\Sigma}$, of this ring is obtained by replacing $H^1(X)$ by $H^1(X, \Sigma)$ in the definition (so that $R_{X,\Sigma} = R_X$ is \mathbb{Z} or $\mathbb{Z}/2$ for the reduced forms of the invariants).

The formal sum

$$\underline{S}_X = \sum_{\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X)} S_X(\mathfrak{s}) \cdot \mathfrak{s}$$

can be considered as an element of the principal (affine) module $R_X[\text{Spin}^{\mathbb{C}}(X)]$ over the group ring $R_X[H^2(X)]$.

Remark. All the constructions and the results obtained below for SW and OS invariants concern in fact any function S_X satisfying a few basic properties of SW and OS invariants, namely A1–A5 formulated in §3.

1.3. Definition of the relative invariant $S_{X,\Sigma}$. Let $\text{Spin}^{\mathbb{C}}(X, \Sigma)$ denote the set of relative $\text{Spin}^{\mathbb{C}}$ structures and $abs_{X,\Sigma}: \text{Spin}^{\mathbb{C}}(X, \Sigma) \rightarrow \text{Spin}^{\mathbb{C}}(X)$ the forgetful map. Gluing of relative $\text{Spin}^{\mathbb{C}}$ structures in a fiber sum $X_+ \#_{\Sigma} X_-$ (see 2.11) yields

$$\vee: \text{Spin}^{\mathbb{C}}(X_+, \Sigma) \times \text{Spin}^{\mathbb{C}}(X_-, \Sigma) \rightarrow \text{Spin}^{\mathbb{C}}(X_+ \#_{\Sigma} X_-, \Sigma), \quad (\mathfrak{r}_+, \mathfrak{r}_-) \mapsto \mathfrak{r}_+ \vee \mathfrak{r}_-$$

whose composition with the forgetful map gives

$$\#_{\Sigma}: \text{Spin}^{\mathbb{C}}(X_+, \Sigma) \times \text{Spin}^{\mathbb{C}}(X_-, \Sigma) \rightarrow \text{Spin}^{\mathbb{C}}(X_+ \#_{\Sigma} X_-), \quad (\mathfrak{r}_+, \mathfrak{r}_-) \mapsto \mathfrak{r}_+ \#_{\Sigma} \mathfrak{r}_-.$$

Choose any relatively minimal Lefschetz fibration $W \rightarrow S^2$ with a fiber Σ and $b_1(W) = 0$ and denote by $\mathfrak{r}_{W,\Sigma} \in \text{Spin}^{\mathbb{C}}(W, \Sigma)$ its canonical relative $\text{Spin}^{\mathbb{C}}$ structure of the Lefschetz fibration introduced in 2.7. If $\Sigma^2 = 0$, then we define for any $\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$

$$S_{X,\Sigma}(\mathfrak{r}) = S_{X\#_{\Sigma}W}(\mathfrak{r}\#_{\Sigma}\mathfrak{r}_{W,\Sigma})$$

If $\Sigma^2 > 0$, then we blow up X at points of Σ to obtain \hat{X} , with $\Sigma^2 = 0$, and let

$$S_{X,\Sigma}(\mathfrak{r}) = S_{\hat{X},\Sigma}(\hat{\mathfrak{r}})$$

where $\hat{\mathfrak{r}}$ is the image of \mathfrak{r} under the natural map $\text{Spin}^{\mathbb{C}}(X, \Sigma) \rightarrow \text{Spin}^{\mathbb{C}}(\hat{X}, \Sigma)$ (see 2.10).

We will let

$$\underline{S}_{X,\Sigma} = \sum_{\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X,\Sigma)} S_{X,\Sigma}(\mathfrak{r}) \cdot \mathfrak{r} \in R_{X,\Sigma}[\text{Spin}^{\mathbb{C}}(X, \Sigma)]$$

Remarks.

- (1) Note that $b_2^+(X\#_{\Sigma}W) > 1$, so $S_{X\#_{\Sigma}W}$ is well defined.
- (2) The differential type of $X\#_{\Sigma}W$ may depend in principle on the framing of Σ in X and W , so proving that $S_{X,\Sigma}$ is independent of W implies also independence of the framings.
- (3) In the case $S_X = SW_X$ we should take care of the homology orientation for $X\#_{\Sigma}W$. It is determined by the given homology orientation of X and the canonical symplectic homology orientation of W (defined in [T4]) following the rule described in [MST] after a modification, which is just such an alternation of the homology orientation which eliminates the sign $(-1)^{b(M,N)}$ that appears in the product formula of [MST]. In the other words, with such a homology orientation the product formula will look like A4, in §3 below. An advantage of such a homology orientation in the fiber sums is that it is preserved by the natural diffeomorphisms $X\#_{\Sigma}Y \cong Y\#_{\Sigma}X$ and $(X\#_{\Sigma}Y)\#_{\Sigma}Z \cong X\#_{\Sigma}(Y\#_{\Sigma}Z)$. In the case of symplectic pairs (X, Σ) and (Y, Σ) , the symplectic homology orientations in X and Y induce the symplectic homology orientation of $X\#_{\Sigma}Y$.

If $\Sigma^2 > 0$, then we choose the homology orientation of \hat{X} induced by that of X .

1.4. The properties of $S_{X,\Sigma}$.

1.4.1. Theorem. *The invariant $S_{X,\Sigma}: \text{Spin}^{\mathbb{C}}(X, \Sigma) \rightarrow R_{X,\Sigma}$ is well-defined as it is independent of the choice of W and has the following properties.*

- (1) *Finiteness of the set $B_{X,\Sigma} = \{\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X, \Sigma) \mid S_{X,\Sigma}(\mathfrak{r}) \neq 0\}$.*
- (2) *The blow-up relation $S_{\hat{X},\Sigma}(\hat{\mathfrak{r}}) = S_{X,\Sigma}(\mathfrak{r})$ (if $\Sigma^2 > 0$ in X).*
- (3) *The conjugation symmetry $S_{X,\Sigma} = \pm S_{X,-\Sigma} \circ \text{conj}$, where $-\Sigma$ is Σ with the opposite orientation, and $\text{conj}: \text{Spin}^{\mathbb{C}}(X, \Sigma) \rightarrow \text{Spin}^{\mathbb{C}}(X, -\Sigma)$ the conjugation involution.*
- (4) *Normalization: if (X, Σ) is a symplectic pair, $\Sigma^2 \geq 0$, and $\mathfrak{r}_{X,\Sigma} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$ the canonical $\text{Spin}^{\mathbb{C}}$ structure of a symplectic manifold, then $S_{X,\Sigma}(\mathfrak{r}_{X,\Sigma}) = 1$ (with respect to the canonical homology orientation of a symplectic manifold). If moreover X admits a relatively minimal Lefschetz pencil with a fiber Σ , then $\mathfrak{r}_{X,\Sigma}$ is the only basic relative $\text{Spin}^{\mathbb{C}}$ structure in $\text{Spin}^{\mathbb{C}}(X, \Sigma)$.*

(5) *Splitting formula relating the absolute and the relative SW invariants:*

$$S_X(\mathfrak{s}) = \sum_{r \in \text{abs}_{X,\Sigma}^{-1}(\mathfrak{s})} S_{X,\Sigma}(\mathfrak{r}),$$

for any $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X)$ such that $c_1(\mathfrak{s})[\Sigma] = \chi(\Sigma) + \Sigma^2$.

(6) *The product formula for a fiber sum $X = X_+ \#_{\Sigma} X_-$ says:*

$$\underline{S}_{X,\Sigma} = (\underline{S}_{X_+,\Sigma})(\underline{S}_{X_-,\Sigma})$$

More explicitly, this means that for any $\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$

$$S_{X,\Sigma}(\mathfrak{r}) = \sum_{\mathfrak{r}_+ \vee \mathfrak{r}_- = \mathfrak{r}} S_{X_+,\Sigma}(\mathfrak{r}_+) S_{X_-,\Sigma}(\mathfrak{r}_-)$$

where \mathfrak{r}_{\pm} are varying in $\text{Spin}^{\mathbb{C}}(X_{\pm}, \Sigma)$. Equivalently, one can write it as

$$S_{X,\Sigma}(\mathfrak{r}_+ \vee \mathfrak{r}_-) = \sum_{k \in \mathbb{Z}} S_{X_+,\Sigma}(\mathfrak{r}_+ + k\sigma_+) S_{X_-,\Sigma}(\mathfrak{r}_- - k\sigma_-)$$

where $\sigma_{\pm} \in H^2(X_{\pm}, \Sigma)$ is dual to the fundamental class of Σ shifted inside $X_{\pm} \setminus \Sigma$.

(7) *Adjunction inequality for $\mathfrak{r} \in B_{X,\Sigma}$ and a membrane $F \subset X$, with the connected complement $\Sigma \setminus \partial F$ (note that positivity of F^2 is not required)*

$$-\chi(F) \geq F^2 + |\mathfrak{r}[F]|.$$

The relations (4) and (5) of Theorem 1.4.1 imply together

Corollary 1.4.2. *For any $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X)$ such that $c_1(\mathfrak{s})[\Sigma] = \chi(\Sigma) + \Sigma^2$*

$$S_X(\mathfrak{s}) = \sum_{\mathfrak{r}_+ \#_{\Sigma} \mathfrak{r}_- = \mathfrak{s}} S_{X_+,\Sigma}(\mathfrak{r}_+) S_{X_-,\Sigma}(\mathfrak{r}_-)$$

Remarks.

- (1) The sign “ \pm ” in Theorem 1.4.1(3) is $(-1)^{\frac{1}{4}(\sigma(X) + \chi(X))}$, like for the absolute invariant S_X .
- (2) The product of $\underline{S}_{X_{\pm},\Sigma}$ in Theorem 1.4.1(6), is induced by the natural affine map

$$\text{Spin}^{\mathbb{C}}(X_+, \Sigma) \times \text{Spin}^{\mathbb{C}}(X_-, \Sigma) \cong \text{Spin}^{\mathbb{C}}(X, M) \rightarrow \text{Spin}^{\mathbb{C}}(X, \Sigma)$$

associated with the corresponding cohomology homomorphisms.

- (3) If $b_2^+(X) = 1$, then the splitting formula 1.4.1(5) should be applied to $SW_X(\mathfrak{s})$ defined by fixing the $[\Sigma]$ -positive orientation of the line $H_2^+(X)$, and $OS_X(\mathfrak{s})$ is defined by fixing the line spanned by $[\Sigma]$ (these are the choices involved into the corresponding product formulae).
- (4) Theorem 1.4.1 does not mention the straightforward properties of the invariants $S_{X,\Sigma}$ which do not involve Σ , for example, the adjunction inequality for a closed

surface $F \subset X \setminus \Sigma$, the blowup formula at a point $x \notin \Sigma$, and the product formula with respect to an additional surface Σ' in $X_{\pm} \setminus \Sigma$. All these properties are formulated exactly like in the case of the absolute invariants S_X and proved by consideration $S_{X \#_{\Sigma} W}$.

- (5) The invariant $S_{X, \Sigma}$ can be defined similarly for a multi-component surface Σ , as one can take fiber sums with auxiliary Lefschetz fibrations along all the components of Σ . The properties of such invariants are analogous to those formulated in Theorem 1.4.1, and the proofs just repeat the arguments in §4.
- (6) The case of genus $g = 1$ is a bit special, mainly because the product formula in this case looks different. Nevertheless, the same definition for $S_{X, \Sigma}$ can be given for $g = 1$, and all the properties except the splitting formula (5) in Theorem 1.4.1 still hold. In the case of $S_X = SW_X$, this follows from the results of Taubes [T5], except for the property (7) (not discussed in [T5]), which is proved by the same arguments as in the case $g > 1$.

1.5. Application: the genus estimate for membranes. By definition, a membrane on a surface Σ in X is a compact surface $F \subset X$ with the boundary $\partial F = F \cap \Sigma$, at no point of which F is tangent to Σ . The self-intersection index F^2 is defined with respect to the normal framing along ∂F which is tangent to Σ . The number $\mathfrak{r}[F]$ (evaluation of $\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$ on F) is defined in 2.6. Throughout the paper we suppose that membranes are connected and oriented, although the adjunction inequality holds as well for disconnected membranes, which follows from additivity of $\chi(F)$, F^2 and $\mathfrak{r}[F]$.

We say that membranes F_1, F_2 on $\Sigma \subset X$ are equivalent, $F_1 \sim F_2$, if F_2 is isotopic in the class of membranes on Σ to F_1' , homologous to F_1 outside Σ , that is, $F_1 - F_1'$ bounds an oriented 3-chain in $X \setminus \Sigma$ (in particular, F_1 and F_1' coincide near their common boundary). It is not difficult to show that F^2 and $\mathfrak{r}[F]$ are invariants of such a homology equivalence relation. In particular, this implies the following minimal genus property for symplectic and Lagrangian membranes in symplectic manifolds.

1.5.1. Corollary. *Assume that X is a symplectic 4-manifold, $\Sigma \subset X$ is an essential surface with $\Sigma^2 \geq 0$, $g(\Sigma) > 1$, and $F \subset X$ is a membrane on Σ with the connected complement $\Sigma \setminus L$ of the boundary $L = \partial F$. Assume furthermore that either Σ is symplectic and F is Lagrangian, or vice versa, Σ is Lagrangian and F is symplectic. Then for another membrane, $F' \sim F$, we have $g(F') \geq g(F)$.*

Proof. The adjunction inequality 1.4.1(7) becomes an equality for such X, Σ, F , and for the canonical $\text{Spin}^{\mathbb{C}}$ structure $\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$ (symplectic or Lagrangian, depending on the case considered). The assumptions formulated for F' imply that $(F')^2 = F^2$ and $\mathfrak{r}[F'] = \mathfrak{r}[F]$, so 1.4.1(7) yields the required estimate for $g(F')$. \square

Another application of 1.4.1(7) is orthogonality of the relative basic classes, $\mathfrak{r} \in B_{X, \Sigma}$, to all (-1) -disc membranes.

1.5.2. Corollary. *If $D \subset X$ is a (-1) -disc membrane on a surface $\Sigma \subset X$ as above, then $\mathfrak{r}[D] = 0$ for all $\mathfrak{r} \in B_{X, \Sigma}$. \square*

1.6. On the calculation of the invariants $S_{X, \Sigma}$. The first trivial observation is vanishing of the relative invariants $S_{X, \Sigma}$ if Σ is not a minimal genus surface in its homology class, because non-vanishing would contradict to the adjunction inequality applied in $X \#_{\Sigma} W$ to the surface Σ' homologous to Σ but of a smaller genus. This argument fails if

Σ' cannot be made disjoint from Σ in X (although the author does not know any example, in which it really cannot).

The simplest example of calculation with non-vanishing invariants $S_{X,\Sigma}$ is contained in Theorem 1.4.1(4). Performing rim-surgery of $\Sigma \subset X$ considered in [FS2] we can get more sophisticated examples, which show possibility of a non-trivial splitting of the basic classes and of the invariants S_X . The invariants $SW_{X,\Sigma}$ implicitly appeared in the work [FS2], as they distinguish the embeddings of surfaces obtained by the rim-surgery from Σ . Moreover, an obvious interpretation of the results of [FS2] yields calculation of the invariant $SW_{X,\Sigma}$ after a rim knot surgery: it is just multiplied by the Alexander polynomial of the knot. More precisely, assume that $\ell \subset \Sigma$ is a simple closed curve, $K \subset S^3$ is a knot and $\Sigma_{K,\ell} \subset X$ is a surface obtained from $\Sigma \subset X$ by rim surgery along ℓ using K as a pattern.

1.6.1. Theorem. $\underline{SW}_{X,\Sigma_{K,\ell}} = \Delta_K(\delta([\ell]^*)) \underline{SW}_{X,\Sigma}$. \square

Here $[\ell]^* \in H^1(\Sigma)$ is dual to $[\ell] \in H_1(\Sigma)$, δ is the boundary map $H^1(\Sigma) \rightarrow H^2(X, \Sigma)$, Δ_K is the Alexander polynomial in the symmetrized form, and $\Delta_K(\delta([\ell]^*))$ is considered as an element of the group ring $\mathbb{Z}[H^2(X, \Sigma)] \subset R_{X,\Sigma}[H^2(X, \Sigma)]$.

1.7. The structure of the paper. In §2 we set up the notation and discuss the properties of the fiber sum and the conjugation operations involving relative $\text{Spin}^{\mathbb{C}}$ structures (this calculus is used in §4).

In §3, we recall some fundamental properties of SW and OS invariants formulating them as axioms A1–A5 sufficient to construct the relative versions of the invariants. The product formula A4 is essentially a reformulation of the formula in [MST] in a bit more general setting. The normalization property A3(3) may be not as known as the other axioms (at least, the author could not find it in literature), along with its corollary 3.1.2, which may be of independent interest. In section §4, we deduce Theorem 1.4.1.

In §5, we discuss some further developments of the invariant $S_{X,\Sigma}$. In 5.1, we discuss possibility to use symplectic pairs (W, Σ) instead of Lefschetz pairs in the definition of $S_{X,\Sigma}$. In 5.2, we consider a version, $S_{X,\Sigma,K}$, of the invariant $S_{X,\Sigma}$ that depends on a subgroup $K \subset H^1(\Sigma)$, and which appears as we relax the assumption $b_1(W) = 0$ in the definition of $S_{X,\Sigma}$ (this may suggest an analogy with the invariant in [CW], which also involves K). To demonstrate a potential of our construction to obtain a refinement of the SW and the OS invariants in the case of a 4-manifold with non-empty boundary, we sketch in 5.3 a relative version of the invariant $F_{X,\mathfrak{s}}^{\text{mix}}$ from [OS3] (for simplicity we consider only the case of OS invariants).

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§2. ABSOLUTE AND RELATIVE $\text{Spin}^{\mathbb{C}}$ STRUCTURES

2.1. Absolute $\text{Spin}^{\mathbb{C}}$ structures. A $\text{Spin}^{\mathbb{C}}$ structure in a principle SO_n bundle, $P \rightarrow X$, is an isomorphism class of $\text{Spin}_n^{\mathbb{C}}$ -extensions, $S \rightarrow P$, of P . The set of $\text{Spin}^{\mathbb{C}}$ structures, $\text{Spin}^{\mathbb{C}}(P)$, has a natural action $\text{Spin}^{\mathbb{C}}(P) \times H^2(X) \rightarrow \text{Spin}^{\mathbb{C}}(P)$, $(\mathfrak{s}, h) \mapsto \mathfrak{s} + h$, which makes it an affine space over $H^2(X)$. The projection $\text{Spin}_n^{\mathbb{C}} \rightarrow \text{SO}_n \times U_1 \rightarrow U_1$ associates

to S its determinant U_1 -bundle, $\det S$, with the Chern class $c_1(S) = c_1(\det S)$, so that $c_1(\mathfrak{s} + h) = c_1(\mathfrak{s}) + 2h$.

We simplify the notation and write just $\text{Spin}^{\mathbb{C}}(X)$ instead of $\text{Spin}^{\mathbb{C}}(P)$, if a principal bundle $P \rightarrow X$ is associated with the tangent bundle TX of a manifold X (an Euclidian structure in TX should be fixed, but its choice is not essential).

2.2. The conjugation involution. The conjugate, $\bar{S} \rightarrow X$, to a principal $\text{Spin}_n^{\mathbb{C}}$ bundle, $S \rightarrow X$, set-theoretically coincides with the latter, but has the conjugate action of $\text{Spin}_n^{\mathbb{C}}$ (induced by the conjugation automorphism in $\text{Spin}_n^{\mathbb{C}}$, which covers the direct product automorphism of $\text{SO}_n \times U_1$, identical on SO_n and non-identical in U_1). The conjugation defines an involution, $\text{conj}_P: \text{Spin}^{\mathbb{C}}(P) \rightarrow \text{Spin}^{\mathbb{C}}(P)$, $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$, such that $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$ and $\overline{\mathfrak{s} + h} = \bar{\mathfrak{s}} - h$ for any $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(P)$ and $h \in H^2(X)$.

2.3. Homology interpretation of $\text{Spin}^{\mathbb{C}}$ structures. It is convenient to identify the set $\text{Spin}^{\mathbb{C}}(P)$ with the coset of the image of $H^2(X)$ under the monomorphism $\pi_P^*: H^2(X) \rightarrow H^2(P)$. This makes transparent the nature of the affine structure in $\text{Spin}^{\mathbb{C}}(P)$. Namely, a $\text{Spin}^{\mathbb{C}}$ extension $F: S \rightarrow P$ can be viewed as a principal U_1 -bundle over P since $\ker(\text{Spin}_n^{\mathbb{C}} \rightarrow \text{SO}_n) \cong U_1$, and the Chern class $c_1(F) \in H^2(P)$ defines the correspondence between $\text{Spin}^{\mathbb{C}}$ structures and those cohomology classes which have a non-trivial restriction $H^2(P) \rightarrow H^2(\text{SO}_n) \cong \mathbb{Z}/2$, $n \geq 3$, to a fiber of P .

Given $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(P) \subset H^2(P)$, one can observe that $\pi_P^*(c_1(\mathfrak{s})) = 2\mathfrak{s}$ and that $\bar{\mathfrak{s}} = -\mathfrak{s}$.

2.4. The canonical $\text{Spin}^{\mathbb{C}}$ structure of a Lefschetz fibration. An almost complex structure in a SO_{2n} bundle defines the $\text{Spin}^{\mathbb{C}}$ -extension associated to the natural homomorphism $U_n \rightarrow \text{Spin}_n^{\mathbb{C}}$. In particular, a symplectic manifold carries a canonical $\text{Spin}^{\mathbb{C}}$ structure represented by the *symplectic $\text{Spin}^{\mathbb{C}}$ extension*, $S \rightarrow X$. It is well-known that the total space X of a Lefschetz fibration $p: X \rightarrow S^2$, carries a compatible symplectic structure (except the case of null-homologous fibers of genus 1, in which X is still almost complex) which gives the associated canonical $\text{Spin}^{\mathbb{C}}$ structure.

In fact, to define a $\text{Spin}^{\mathbb{C}}$ structure in a vector bundle $E \rightarrow X$ it is sufficient to have an almost complex structure over its 3-skeleton, $\text{Ske}_3 X$, only (more precisely, $\text{Spin}^{\mathbb{C}}$ structures can be viewed as equivalence classes of those almost complex structures over $\text{Ske}_2 X$ that can be extended to $\text{Ske}_3 X$). This gives an alternative way to introduce the canonical $\text{Spin}^{\mathbb{C}}$ structure in the Lefschetz fibration, using $U_1 \times U_1$ reduction of the tangent bundle τ_X in the complement of the critical point set of p determined by the “vertical” and the complementary “horizontal” $\text{SO}_2 = U_1$ subbundles.

2.5. Relative $\text{Spin}^{\mathbb{C}}$ -structures. In the definition of a relative structure, in addition to the $\text{Spin}_n^{\mathbb{C}}$ -bundle $S \rightarrow X$ considered in 2.1, we fix an isomorphism between the restriction $S|_A$ and a certain reference principal $\text{Spin}^{\mathbb{C}}$ bundle, $S_A \rightarrow A$. Such a reference bundle appears naturally for example if X is a 4-manifold and A is a surface $\Sigma \subset X$, or a tubular neighborhood N of Σ , or the boundary ∂N , since in these cases $\tau_X|_A$ admits a natural $U_1 \times U_1$ reduction and thus, the associated $\text{Spin}_4^{\mathbb{C}}$ -extension.

More formally speaking, let (X, A) be a CW-pair, $\pi_P: P \rightarrow X$ a principal SO_n bundle, $\pi_{P|A}: P_A \rightarrow A$ the restriction of π_P over A , and $F_A: S_A \rightarrow P_A$ a $\text{Spin}^{\mathbb{C}}$ extension of P_A . A *relative $\text{Spin}^{\mathbb{C}}$ extension of P* with respect to S_A is a $\text{Spin}^{\mathbb{C}}$ -extension, $S \rightarrow X$, $F: S \rightarrow P$, together with an isomorphism of $\text{Spin}^{\mathbb{C}}$ extensions over A , that is $R: S|_A \rightarrow S_A$, such that $F_A \circ R$ is the restriction of F over A . An isomorphism between relative $\text{Spin}^{\mathbb{C}}$ -extensions

$F^{(i)}: S^{(i)} \rightarrow P$, $R^{(i)}: S^{(i)}|_A \rightarrow S_A$, $i = 1, 2$, is defined as an isomorphism $S^{(1)} \rightarrow S^{(2)}$ of $\text{Spin}^{\mathbb{C}}$ bundles whose restriction over A commutes with $R^{(1)}$ and $R^{(2)}$. An isomorphism class of relative $\text{Spin}^{\mathbb{C}}$ -extensions is called a *relative $\text{Spin}^{\mathbb{C}}$ structure*, and the set of such structures is denoted by $\text{Spin}^{\mathbb{C}}(P, S_A)$, or simply by $\text{Spin}^{\mathbb{C}}(X, A)$ if P and S_A are evident.

It is straightforward to check that $\text{Spin}^{\mathbb{C}}(P, S_A)$ is an affine space over $H^2(X, A)$ and the natural forgetful map $\text{abs}: \text{Spin}^{\mathbb{C}}(P, S_A) \rightarrow \text{Spin}^{\mathbb{C}}(P)$ is affine with respect to the cohomology forgetful homomorphism $H^2(X, A) \rightarrow H^2(X)$.

The conjugation involution defined like in the absolute case interchanges $\text{Spin}^{\mathbb{C}}(P, S_A)$ with $\text{Spin}^{\mathbb{C}}(P, \overline{S}_A)$. It is anti-affine, that is $\overline{\mathfrak{r} + h} = \bar{\mathfrak{r}} - h$ for $\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(P, S_A)$, $h \in H^2(X, A)$.

2.6. Relative $\text{Spin}^{\mathbb{C}}$ structures with respect to surfaces, $\Sigma \subset X$, and their evaluation on membranes. Let $S_{\Sigma} \rightarrow \Sigma$ denote the canonical $\text{Spin}^{\mathbb{C}}$ extension defined by the $U_1 \times U_1$ reduction due to the splitting of the tangent bundle $\tau_X|_{\Sigma}$ along Σ into a sum $\tau_{\Sigma} \oplus \nu_{\Sigma}$ of the tangent and the normal bundles to Σ . Note that the inversion of the orientation of Σ results in the conjugation of the associated canonical $\text{Spin}_4^{\mathbb{C}}$ bundle, $S_{-\Sigma} = \overline{S}_{\Sigma}$. In particular, the conjugation involution in this case is $\text{Spin}^{\mathbb{C}}(X, \Sigma) \rightarrow \text{Spin}^{\mathbb{C}}(X, -\Sigma)$.

Assume that $\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$. Note that any membrane $(F, \partial F) \subset (X, \Sigma)$ defines trivializations of the both τ_{Σ} and ν_{Σ} along ∂F , and thus provides a trivialization of the determinant bundle $\det S_{\Sigma} \cong \tau_{\Sigma} \otimes \nu_{\Sigma}$. The obstruction class in $H^2(F, \partial F)$ for extension of this trivialization to the whole F , as it is evaluated on the fundamental class, $[F, \partial F]$, gives an integer denoted by $\mathfrak{r}[F]$. It is easy to observe that $\bar{\mathfrak{r}}[F] = \mathfrak{r}[-F] = -(\mathfrak{r}[F])$.

2.7. The canonical relative $\text{Spin}^{\mathbb{C}}$ structures in the case of symplectic or Lagrangian surface, $\Sigma \subset X$. Assume now that $\Sigma \subset X$ is a symplectic surface with respect to some symplectic structure ω in X that is $\omega|_{\Sigma} > 0$. Then we can define the canonical *symplectic relative $\text{Spin}^{\mathbb{C}}$ structure*, $\mathfrak{r}_{X, \Sigma} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$ whose image $\text{abs}(\mathfrak{r}_{X, \Sigma}) \in \text{Spin}^{\mathbb{C}}(X)$ is the absolute symplectic canonical $\text{Spin}^{\mathbb{C}}$ structure introduced in section 2.4. Namely, the structure $\mathfrak{r}_{X, \Sigma}$ is represented by a $\text{Spin}^{\mathbb{C}}$ extension $S \rightarrow P$ of the principal SO_4 bundle $P \rightarrow X$, which arises from an almost complex structure determined in τ_X as we fix a Riemannian metric in X compatible with ω . If we choose such a metric making the surface Σ pseudo-holomorphic (which is always possible), then the restriction $S|_{\Sigma}$ is naturally identified with the canonical $\text{Spin}^{\mathbb{C}}$ -bundle $S_{\Sigma} \rightarrow \Sigma$.

Consider now the case of a Lagrangian surface Σ in a symplectic manifold X , in which we can similarly define the canonical *Lagrangian relative $\text{Spin}^{\mathbb{C}}$ structure*, $\mathfrak{r}_{X, \Sigma} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$. One way to do it is to make a Lagrangian surface symplectic by a perturbation of the symplectic form. There is also an alternative description of this structure: it concerns also the exceptional case of a null-homologous torus, in which such a perturbation is impossible, and makes it evident that the choice of a perturbation is not essential. It involves the canonical isomorphism between the two different almost complex structure in $\tau_X|_{\Sigma}$: the one induced from X that is coming from the isomorphism $\tau_X|_{\Sigma} \cong \tau_{\Sigma} \otimes \mathbb{C}$, and the one arising from $U_1 \times U_1$ reduction due to the splitting $\tau_X|_{\Sigma} \cong \tau_{\Sigma} \oplus \nu_{\Sigma}$ (this isomorphism is just a special case of the canonical one, $\xi \otimes \mathbb{C} \cong \xi \oplus \xi$, for any complex bundle ξ). The induced isomorphism of the associated $\text{Spin}^{\mathbb{C}}$ bundles covers an automorphism of $\tau_X|_{\Sigma}$ that can be canonically connected to the identity by an isotopy. This gives an isomorphism between these $\text{Spin}^{\mathbb{C}}$ extensions, which defines the Lagrangian relative $\text{Spin}^{\mathbb{C}}$ structure.

Remark. Note that the canonical $\text{Spin}^{\mathbb{C}}$ structure $\mathfrak{r}_{X,\Sigma} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$ of a symplectic pair (X, Σ) is invariant under the monodromy induced by any symplectic isotopy of Σ in X , whereas any other structure, $\mathfrak{r} = \mathfrak{r}_{X,\Sigma} + h \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$, $h \in H^2(X, \Sigma)$, is sent by the monodromy to $\mathfrak{r}_{X,\Sigma} + f_*(h)$, where f_* is the cohomology monodromy.

The same concerns Lagrangian surfaces and Lagrangian isotopy.

2.8. Lefschetz fibrations and their conjugates. A special case of our interest is Σ being a fiber of a Lefschetz fibration $p: X \rightarrow S^2$, (or more generally, a fiber in a Lefschetz pencil). Such a fiber $\Sigma \subset X$ is symplectic with respect to the symplectic form ω supported by the Lefschetz fibration (or pencil), so there is a canonical structure $\mathfrak{r}_{X,\Sigma} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$ from section 2.7.

The conjugate Lefschetz fibration $\bar{p}: \bar{X} \rightarrow \bar{S}^2$ is by definition, set-theoretically the same as p , however, with the opposite orientation chosen in the base-space $\bar{S}^2 = -S^2$ and in the fibers, $\bar{\Sigma} = -\Sigma$ (so that \bar{X} itself has the same orientation as X). It is not difficult to observe that $\overline{\mathfrak{r}_{X,\Sigma}} = \mathfrak{r}_{\bar{X},\bar{\Sigma}} \in \text{Spin}^{\mathbb{C}}(X, -\Sigma)$.

2.9. The excision and the homotopy invariance theorems for $\text{Spin}^{\mathbb{C}}$ structures. The propositions stated below mimic the standard results for the cohomology and follow automatically from the latter, since an affine map associated with an isomorphism must be an affine isomorphism.

2.9.1. Proposition (excision). Assume that a CW complex Z is decomposed into a union of subcomplexes, $Z = X \cup Y$, $A = X \cap Y$. Consider a principal SO_n bundle $P_Z \rightarrow Z$ and let P_X, P_Y, P_A denote its restrictions over X, Y , and A respectively. Fix a $\text{Spin}^{\mathbb{C}}$ extension, $F_Y: S_Y \rightarrow P_Y$ and let $F_A: S_A \rightarrow P_A$ denote its restriction over A . Then the restriction map

$$\text{Spin}^{\mathbb{C}}(P_Z, S_Y) \rightarrow \text{Spin}^{\mathbb{C}}(P_X, S_A)$$

is an isomorphism of affine spaces agreeing with the isomorphism $H^2(Z, Y) \cong H^2(X, A)$. \square

2.9.2. Proposition (homotopy invariance). Assume that Σ is a deformation retract of $N \subset X$. Let $F_N: S_N \rightarrow P_N$ be a $\text{Spin}^{\mathbb{C}}$ extension and $S_{\Sigma} = S_N|_{\Sigma}$. Then the restriction map $\text{Spin}^{\mathbb{C}}(P_X, S_N) \rightarrow \text{Spin}^{\mathbb{C}}(P_X, S_{\Sigma})$ is an isomorphism of the affine spaces agreeing with the isomorphism $H^2(X, N) \cong H^2(X, \Sigma)$. \square

2.9.3. Corollary. Let $\Sigma \subset X$ be a surface in a 4-manifold, $N \subset X$ its compact tubular neighborhood, $M = \partial N$, and $X^{\circ} = X \setminus \text{Int}(N)$. Then we have canonical affine isomorphisms $\text{Spin}^{\mathbb{C}}(X, \Sigma) \cong \text{Spin}^{\mathbb{C}}(X, N) \cong \text{Spin}^{\mathbb{C}}(X^{\circ}, M)$. \square

2.10. Connected sums and blowing up of $\text{Spin}^{\mathbb{C}}$ structures. Definitions of the connected sum and the blowup operations for the absolute $\text{Spin}^{\mathbb{C}}$ structures are quite trivial and well-known. They can be also easily extended to relative $\text{Spin}^{\mathbb{C}}$ structures (we leave it as an exercise). Namely, for a pair of n -manifolds X_{\pm} , with codimension 2 submanifolds Σ_{\pm} , and structures $\mathfrak{r}_{\pm} \in \text{Spin}^{\mathbb{C}}(X_{\pm}, \Sigma_{\pm})$, one can define the connected sum $\mathfrak{r}_+ \# \mathfrak{r}_- \in \text{Spin}^{\mathbb{C}}(X_+ \# X_-, \Sigma_+ \# \Sigma_-)$, where $\Sigma_+ \# \Sigma_- \subset X_+ \# X_-$ is the internal connected sum of (X_{\pm}, Σ_{\pm}) .

Let $\Sigma \subset X$ be a surface in a four-manifold, and $\hat{\Sigma} \subset \hat{X}$ its proper image after blowing up X at a point of Σ , that is $\Sigma \# \mathbb{CP}^1 \subset X \# (-\mathbb{CP}^2)$. For $\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$, we define $\hat{\mathfrak{r}} \in \text{Spin}^{\mathbb{C}}(\hat{X}, \hat{\Sigma})$ as $\hat{\mathfrak{r}} = \mathfrak{r} \# \mathfrak{r}_{-1}$, where $\mathfrak{r}_{-1} \in \text{Spin}^{\mathbb{C}}(-\mathbb{CP}^2, \mathbb{CP}^1)$ is the unique structure such that $c_1(\text{abs}(\mathfrak{r}_{-1})) = -1$.

2.11. Fiber sums of relative $\text{Spin}^{\mathbb{C}}$ structures. Let Σ be a closed oriented surface of genus $g > 1$. We say that X is a Σ -marked 4-manifold, if there is a fixed smooth embedding $f: \Sigma \rightarrow X$ endowed with a normal framing of $f(\Sigma)$ (in particular, $\Sigma^2 = 0$). To simplify the notation, we will be writing $\text{Spin}^{\mathbb{C}}(X, \Sigma)$ rather than $\text{Spin}^{\mathbb{C}}(X, f(\Sigma))$.

Given Σ -marked 4-manifolds X_{\pm} , consider their fiber sum $X = X_+ \#_{\Sigma} X_- = X_+^{\circ} \cup_f X_-^{\circ}$, where $X_{\pm}^{\circ} = X_{\pm} \setminus \text{Int}(N_{\pm})$ (the complements of the tubular neighborhoods of Σ), and the gluing diffeomorphism $f: \partial N_+ \rightarrow \partial N_-$ is naturally determined by the trivialization of $N_{\pm} \rightarrow \Sigma$ respecting the framings, so that ∂N_+ and $-\partial N_-$ are identified with $M = \Sigma \times S^1$. Note that X has an induced structure of Σ -marked 4-manifold, since $\Sigma_t = \Sigma \times t \subset \Sigma \times S^1$ has a natural normal framing. Operations $\mathfrak{r}_+ \#_{\Sigma} \mathfrak{r}_- \in \text{Spin}^{\mathbb{C}}(X)$ and $\mathfrak{r}_+ \vee \mathfrak{r}_- \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$ for $\mathfrak{r}_{\pm} \in \text{Spin}^{\mathbb{C}}(X_{\pm}, \Sigma)$ are the compositions of the isomorphism

$$\text{Spin}^{\mathbb{C}}(X_+, \Sigma) \times \text{Spin}^{\mathbb{C}}(X_-, \Sigma) \cong \text{Spin}^{\mathbb{C}}(X_+^{\circ}, M) \times \text{Spin}^{\mathbb{C}}(X_-^{\circ}, M) \cong \text{Spin}^{\mathbb{C}}(X, M)$$

with the forgetful maps $\text{Spin}^{\mathbb{C}}(X, M) \rightarrow \text{Spin}^{\mathbb{C}}(X)$ and $\text{Spin}^{\mathbb{C}}(X, M) \rightarrow \text{Spin}^{\mathbb{C}}(X, \Sigma)$.

Given $\mathfrak{s}_{\pm} \in \text{Spin}^{\mathbb{C}}(X_{\pm}, \mathfrak{s}_{\Sigma})$ we denote by $\mathfrak{s}_- \#_{\Sigma} \mathfrak{s}_+$ a subset of $\text{Spin}^{\mathbb{C}}(X)$ consisting of the structures $\mathfrak{r}_- \#_{\Sigma} \mathfrak{r}_+$ for all $\mathfrak{r}_{\pm} \in \text{abs}_{X_{\pm}, \Sigma}^{-1}(\mathfrak{s}_{\pm})$. It is not difficult to check that the set $\mathfrak{s}_- \#_{\Sigma} \mathfrak{s}_+$ is affine with respect to the subgroup $\Delta_M \subset H^2(X)$, which is the image of $H^1(\Sigma)$ under the product of the homomorphism $q^*: H^1(\Sigma) \rightarrow H^1(M)$ induced by the projection $q: M \cong \Sigma \times S^1 \rightarrow \Sigma$ and the boundary map $\delta_M: H^1(M) \rightarrow H^2(X)$.

One can also interpret $\mathfrak{s}_- \#_{\Sigma} \mathfrak{s}_+$ as a set consisting of those $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X)$ which have $d(\mathfrak{s}) = d(\mathfrak{s}_+) + d(\mathfrak{s}_-)$ and whose restriction to X_{\pm}° coincides with that of \mathfrak{s}_{\pm} .

2.12. The natural properties of the fiber sum operations with the relative $\text{Spin}^{\mathbb{C}}$ structures. It is not difficult to check that the operations introduced in 2.11 satisfy the following natural properties

$$\begin{aligned} \mathfrak{r}_1 \vee \mathfrak{r}_2 &= \mathfrak{r}_2 \vee \mathfrak{r}_1, \text{ and thus } \mathfrak{r}_1 \#_{\Sigma} \mathfrak{r}_2 = \mathfrak{r}_2 \#_{\Sigma} \mathfrak{r}_1 \\ (\mathfrak{r}_1 \vee \mathfrak{r}_2) \vee \mathfrak{r}_3 &= \mathfrak{r}_1 \vee (\mathfrak{r}_2 \vee \mathfrak{r}_3), \text{ and thus } (\mathfrak{r}_1 \vee \mathfrak{r}_2) \#_{\Sigma} \mathfrak{r}_3 = \mathfrak{r}_1 \#_{\Sigma} (\mathfrak{r}_2 \vee \mathfrak{r}_3) \\ &\quad (\text{we denote the latter sum } \mathfrak{r}_1 \#_{\Sigma} \mathfrak{r}_2 \#_{\Sigma} \mathfrak{r}_3) \\ \overline{\mathfrak{r}_1 \vee \mathfrak{r}_2} &= \bar{\mathfrak{r}}_1 \vee \bar{\mathfrak{r}}_2, \text{ and thus } \overline{\mathfrak{r}_1 \#_{\Sigma} \mathfrak{r}_2} = \bar{\mathfrak{r}}_1 \#_{-\Sigma} \bar{\mathfrak{r}}_2 \end{aligned}$$

where the equalities mean that the obvious diffeomorphisms

$$\begin{aligned} (X_1 \#_{\Sigma} X_2, \Sigma) &\cong (X_2 \#_{\Sigma} X_1, \Sigma) \\ ((X_1 \#_{\Sigma} X_2) \#_{\Sigma} X_3, \Sigma) &\cong (X_1 \#_{\Sigma} (X_2 \#_{\Sigma} X_3), \Sigma) \\ X_1 \#_{\Sigma} X_2 &\cong X_1 \#_{-\Sigma} X_2 \end{aligned}$$

send one of the corresponding $\text{Spin}^{\mathbb{C}}$ structures to the other.

Some ambiguity in the notion of “the obvious diffeomorphism”, related in particular to the ambiguity in Σ -marking of the fiber sums, turns out to be not essential. It is also straightforward to check the following

Proposition 2.12.1. *Assume that (X_i, Σ) are symplectic pairs, $i = 1, 2$, $X = X_1 \#_{\Sigma} X_2$, and $\mathfrak{r}_i \in \text{Spin}^{\mathbb{C}}(X_i, \Sigma)$ are the canonical relative $\text{Spin}^{\mathbb{C}}$ structures. Then $\mathfrak{r}_1 \vee \mathfrak{r}_2 \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$ is also the canonical $\text{Spin}^{\mathbb{C}}$ structure of the symplectic pair (X, Σ) . In particular, $\mathfrak{r}_1 \#_{\Sigma} \mathfrak{r}_2 \in \text{Spin}^{\mathbb{C}}(X)$ is the canonical symplectic $\text{Spin}^{\mathbb{C}}$ structure of X . \square*

§3. THE BASIC PROPERTIES OF THE ABSOLUTE SW AND OS INVARIANTS

3.1. The axioms. Axioms A1, A3 and A4 below are essential first of all for the definition of $S_{X,\Sigma}$, that is for proving its independence of the choice of W , whereas axioms A2 and A5 are required only for deriving the corresponding properties of $S_{X,\Sigma}$ (namely, 3 and 7) in Theorem 1.4.1. We suppose that the closed 4-manifolds X and X_\pm below have $b_2^+ > 1$ (in the case $b_2^+ = 1$ the formulations are similar, but require a bit more care).

A1. Finiteness. The set of the basic $\text{Spin}^\mathbb{C}$ structures $B_X = \{\mathfrak{s} \in \text{Spin}^\mathbb{C}(X) \mid S_X(\mathfrak{s}) \neq 0\}$ is finite for any X .

A2. Conjugation symmetry. $S_X \circ \text{conj}_X = \pm S_X$, where conj_X is the conjugation involution in $\text{Spin}^\mathbb{C}(X)$.

A3. Normalization. Assume that X is symplectic, endowed with the canonical homology orientation (see [T4]), and $\mathfrak{s}_X \in \text{Spin}^\mathbb{C}(X)$ the canonical $\text{Spin}^\mathbb{C}$ structure. Then

- (1) $S_X(\mathfrak{s}_X) = 1$.
- (2) If $\Sigma \subset X$ is a symplectic surface of genus $g > 1$, $\Sigma^2 = 0$, and a basic structure $\mathfrak{s} \in B_X$ has the same restriction to $X \setminus \Sigma$ as \mathfrak{s}_X , then $\mathfrak{s} = \mathfrak{s}_X$. In the other words, if $\mathfrak{s} = \mathfrak{s}_X + n\sigma \in B_X$, where $\sigma \in H^2(X)$ is dual to Σ , then $n = 0$.
- (3) For any fiber sum $X \#_\Sigma Y = X^\circ \cup Y^\circ$ of a relatively minimal Lefschetz fibration X with any Σ -marked 4-manifold Y , the restriction $\mathfrak{s}|_{X^\circ} \in \text{Spin}^\mathbb{C}(X^\circ)$ of any basic $\text{Spin}^\mathbb{C}$ structure $\mathfrak{s} \in \text{Spin}^\mathbb{C}(X \#_\Sigma Y, \mathfrak{s}_\Sigma)$, coincides with the restriction $\mathfrak{s}_X|_{X^\circ} \in \text{Spin}^\mathbb{C}(X^\circ)$.

A4. Product formula. Let $X = X_- \#_\Sigma X_+ = X_-^\circ \cup X_+^\circ$ be a fiber sum like in 2.11, with a fiber Σ of genus $g > 1$. Choose $\mathfrak{s}_\pm \in \text{Spin}^\mathbb{C}(X_\pm, \mathfrak{s}_\Sigma)$ and let $\sigma_\pm \in H^2(X_\pm)$ denote the Poincare dual class to $\Sigma \subset X_\pm$. Then

$$\sum_{k \in \mathbb{Z}} S_{X_+}(\mathfrak{s}_+ + k\sigma_+) S_{X_-}(\mathfrak{s}_- - k\sigma_-) = \sum_{\mathfrak{s} \in \mathfrak{s}_- \#_\Sigma \mathfrak{s}_+} S_X(\mathfrak{s}).$$

A5. Adjunction inequality: $-\chi(\Sigma) \geq \Sigma^2 + |c_1(\mathfrak{s})[\Sigma]|$, for any $\mathfrak{s} \in B_X$ and a surface $\Sigma \subset X$ of genus $g > 0$, with $\Sigma^2 \geq 0$.

Combining properties A3(1)-(2) with A4 and taking into account the remark about $\mathfrak{s}_- \#_\Sigma \mathfrak{s}_+$ and Δ_M in the end of section 2.11, we obtain the following

3.1.1. Corollary. Assume that the pair (X_-, Σ) in A4 is symplectic, and $\mathfrak{s}_- \in \text{Spin}^\mathbb{C}(X_-)$, $\mathfrak{r}_- \in \text{Spin}^\mathbb{C}(X_-, \Sigma)$ are the canonical absolute and relative $\text{Spin}^\mathbb{C}$ structures. Then for any $\mathfrak{s}_+ \in \text{Spin}^\mathbb{C}(X_+, \mathfrak{s}_\Sigma)$

$$S_{X_+}(\mathfrak{s}_+) = \sum_{\mathfrak{s} \in \mathfrak{s}_- \#_\Sigma \mathfrak{s}_+} S_X(\mathfrak{s}) = \sum_{\mathfrak{r}_+ \in \text{abs}^{-1}(\mathfrak{s}_+)} S_X(\mathfrak{r}_+ \#_\Sigma \mathfrak{r}_-) = \sum_{h \in \Delta_M} S_X(\mathfrak{s} + h)$$

In the last sum, \mathfrak{s} is any fixed element of $\mathfrak{s}_- \#_\Sigma \mathfrak{s}_+ \subset \text{Spin}^\mathbb{C}(X)$ and Δ_M is like in 4.1.

Applying A3(3) in the case of $Y = \Sigma \times S^2$, and making use of A3(2) we obtain the following

3.1.2. Corollary. Assume that $X \rightarrow S^2$ is a relatively minimal Lefschetz fibration with a fiber Σ . Then the canonical structure \mathfrak{s}_X is the only basic structure in the set $\text{Spin}^\mathbb{C}(X, \mathfrak{s}_\Sigma) = \{\mathfrak{s} \in \text{Spin}^\mathbb{C}(X) \mid c_1(\mathfrak{s})[\Sigma] = \chi(\Sigma)\}$.

3.2. Properties A1, A2, and A5.

A1. The finiteness is a fundamental well-known property of SW invariants, which holds as well for OS invariants, see [OS3], Theorem 3.3.

A2. It is also a well-known property. In fact, a set-theoretic identification of the conjugate $\text{Spin}^{\mathbb{C}}$ -bundles S and \bar{S} gives a point-wise correspondence (possibly alternating the orientations) between the solutions spaces to the SW equations associated with S and \bar{S} . For the case of OS invariants, see [OS3], Theorem 3.5.

A5. The original version of inequality of Kronheimer-Mrowka and Taubes concerns also the case of an essential sphere, not important for this paper. In the most general form (including the case of $\Sigma^2 < 0$) the adjunction formula is given in [OS2], Theorems 1.1–1.7 for SW invariants, and in [OS3], Theorem 1.4. for OS invariants.

3.3. Lefschetz normalization properties A3(1)–(3).

A3(1). For SW invariants this property is proved by Taubes [T1] (the Main Theorem) in a more general setting, for symplectic manifolds. The case of OS invariants was considered in [OS4], Theorem 5.1. For this part of A3 the minimality condition is not required.

A3(2). Since the symplectic manifolds are of the simple type (see [T4] and Theorem 02(6)), the structure $\mathfrak{s}_X + n\sigma$ cannot be basic for $n \neq 0$, because $d(\mathfrak{s}_X + n\sigma) - d(\mathfrak{s}_X) = \frac{1}{4}[c_1^2(\mathfrak{s}_X + n\sigma) - c_1^2(\mathfrak{s}_X)][X] = nc_1(\mathfrak{s}_X)[\Sigma] = n\chi(\Sigma) \neq 0$ in the case of $g(\Sigma) > 1$.

A3(3) for OS invariants. It is proved by the arguments in Lemma 5.7 from [OS4] for OS invariants. The same scheme of the proof works for SW invariants as well, so we will briefly review it (sending a reader to [OS4] for the notation and details).

The first step is to observe that the canonical structure $\mathfrak{s}_X \in \text{Spin}^{\mathbb{C}}(X)$ is (up to a summand $n\sigma$) the only one satisfying the adjunction inequality with respect to a certain family of surfaces $F \subset X \setminus \Sigma$. For these surfaces $F^2 < 0$, and so in principle the inequality may fail for a basic structure $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X)$, but in this case there is another basic $\text{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}' = \mathfrak{s} + f$, where $f \in H^2(X)$ is Poincare dual to $[F]$, and there exists $\xi \in \mathbb{A}_F$ such that $\Phi_{X,\mathfrak{s}'}(\xi x) = \Phi_{X,\mathfrak{s}}(x)$ for any $x \in \mathbb{A}_X$ (the action of ξ on x means the action of the image of ξ under the inclusion map $\mathbb{A}_F \rightarrow \mathbb{A}_X$). One can notice next that the construction of surfaces F in [OS4] yields a natural epimorphism $H_1(\Sigma) \rightarrow H_1(F)$ commuting with the inclusion homomorphisms from $H_1(\Sigma)$ and $H_1(F)$ to $H_1(X)$, and so we may assume that $\xi \in \mathbb{A}_{\Sigma}$. The second key observation is triviality of the action of \mathbb{A}_{Σ} in $HF^+(M, \mathfrak{t})$, where $M = \Sigma \times S^1$ and $\mathfrak{t} = \mathfrak{s}|_M$ is the canonical structure induced from $\mathfrak{s}_{\Sigma} \in \text{Spin}^{\mathbb{C}}(\Sigma)$ by the projection $M \rightarrow \Sigma$. This triviality is deduced in [OS4] as an immediate corollary of the isomorphism $HF^+(M, \mathfrak{t}) \cong \mathbb{Z}$.

The third ingredient of the proof is the relation between the invariants $F_{X^{\odot}, \mathfrak{s}'^{\odot}}^{\text{mix}}$ and $\Phi_{X, \mathfrak{s}'}$ (the latter is dual to $OS_X(\mathfrak{s}')$ in our notation), where $X^{\odot} = X \setminus (\text{Int } N \cup \text{Int } B^4)$ is the complement of a tubular neighborhood N of Σ and a ball $B^4 \subset X$ disjoint from N . X^{\odot} is viewed as a cobordism from S^3 to $M = \partial N$, so that $F_{X^{\odot}, \mathfrak{s}'^{\odot}}^{\text{mix}}$ takes values in $HF^+(M, \mathfrak{t}) \cong \mathbb{Z}$. More precisely, $\Phi_{X, \mathfrak{s}'}(\xi x)$ and thus $\Phi_{X, \mathfrak{s}}(x)$, or equivalently, $OS_X(\mathfrak{s})$, vanishes as it is the homogeneous part of $F_{X^{\odot}, \mathfrak{s}'^{\odot}}^{\text{mix}}(\Theta_- \otimes (\xi x)) = \xi F_{X^{\odot}, \mathfrak{s}'^{\odot}}^{\text{mix}}(\Theta_- \otimes x)$, where $\Theta_- \in HF^-(S^3)$ is the generator in the upper dimension, and $\mathfrak{s}'^{\odot} = \mathfrak{s}'|_{X^{\odot}}$ (see the proof of Lemma 5.6 of [OS4]). This contradicts to the assumption that \mathfrak{s} (and thus \mathfrak{s}') is a basic structure.

Applying these arguments to $X \#_{\Sigma} Y$, we conclude similarly that if $OS_{X \#_{\Sigma} Y}(\mathfrak{s}) \neq 0$, for $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X \#_{\Sigma} Y)$ such that $\mathfrak{s}|_{\Sigma} = \mathfrak{s}_{\Sigma}$, then $\mathfrak{s}|_{X^{\odot}}$ is canonical, since otherwise $F_{X^{\odot} \#_{\Sigma} Y, \mathfrak{s}'^{\odot}}^{\text{mix}}$

and, thus, $OS_{X\#_\Sigma Y}(\mathfrak{s})$ vanish. \square

A3(3) for SW invariants. The first step for SW invariants is like for OS invariants, since the generalized adjunction inequalities look similar in the both theories (cf. [OS1], [OS2] and [OS4]). Next, the Seiberg-Witten-Floer homology groups $HF_*^{SW}(M, \mathfrak{t}) \cong \mathbb{Z}$, (see [MW], Theorem 1.7), and so the action of \mathbb{A}_Σ considered in [CW] is trivial on this group for the same reason as in the case of the OS invariants.

The final step goes also like in the OS-theory, but instead of $F_{X^\circ, \mathfrak{s}'^\circ}^{\text{mix}}(\Theta_- \otimes x)$ we consider the function $\phi_{X^\circ}^{SW}(\mathfrak{s}'^\circ, x^\circ)$ from [CW], where $X^\circ = X \setminus \text{Int } N$, $x^\circ = x|_{X^\circ}$ and $\mathfrak{s}'_{X^\circ}^\circ = \mathfrak{s}'|_{X^\circ}$. To deduce vanishing of $SW_{X\#_\Sigma Y}(\xi x)$ we can use the product formula, in Theorem 1.2 of [CW], which implies for $X\#_\Sigma Y = X^\circ \cup Y^\circ$ that

$$SW_{X\#_\Sigma Y, \mathfrak{s}'}(\xi x \otimes y) = \langle [u]\pi_1(\phi_{X^\circ}^{SW}(\mathfrak{s}'_{X^\circ}^\circ, \xi x)), \pi_2(\phi_{Y^\circ}^{SW}(\mathfrak{s}'_{Y^\circ}^\circ, y)) \rangle$$

and we can conclude that the product vanishes, because the first factor vanishes. \square

3.4. The product formula A4. A4 is a reformulation of the well known product formula [MST], Theorem 3.1 (note that the sign changes due to our convention on the homology orientations in Remark 3 of section 1.3). Note only that the formula in [MST] is stated for the version of the SW invariants corresponding to the case of $R_X = \mathbb{Z}[U]$, which is not as general as $R_X = \mathbb{A}_X^*$. It may be discussable if the arguments of [MST] without essential changes are applicable to the most general case, but anyway, after [MST], much more general gluing formulae were established, see for instance Theorem 1.2 in [CW], which concerns 4-manifolds with an arbitrary boundary and contains A4 in the full generality as a corollary.

In the case of OS invariants, A4 can be derived from the product formula [OS3], Theorem 3.4, applied to the fiber sums, although it may look not so obvious as in the case of SW invariants. To clarify it, we give some comments, which are basically extracted from [OS3] and [OS4].

Puncturing a fiber sum, $X = X_- \#_\Sigma X_+ = X_-^\circ \cup X_+^\circ$, at a pair of points, we obtain $X^\circ = X_-^\circ \cup X_+^\circ$, where $X_\pm^\circ = X_\pm^\circ \setminus B^4$. X° can be viewed as a product of cobordisms $X_-^\circ: S^3 \rightarrow M \cong S^1 \times \Sigma$ and $X_+^\circ: M \rightarrow S^3$. The product formula [OS3], Theorem 3.4, says

$$[F_{X_+^\circ, \mathfrak{s}_+^\circ}^+(F_{X_-^\circ, \mathfrak{s}_-^\circ}^{\text{mix}}(\theta_- \otimes x_-) \otimes x_+)]_0 = \sum_{\mathfrak{s}|_{X_\pm^\circ} = \mathfrak{s}_\pm^\circ} \Phi_{X, \mathfrak{s}}(x_- \otimes x_+)$$

where $\mathfrak{s}_\pm^\circ \in \text{Spin}^C(X_\pm^\circ)$, $\mathfrak{s}_\pm^\circ|_M = \mathfrak{t}$, and $[x]_0 \in HF_0^+(S^3)$ denotes the 0-dimensional component of $x \in HF^+(S^3)$.

We assume here that Spin^C structure \mathfrak{s}_\pm° is induced in X_\pm° from $\mathfrak{s}_\pm \in \text{Spin}^C(X_\pm, \mathfrak{s}_\Sigma)$, and, thus, its restriction, \mathfrak{t} is the canonical Spin^C structure determined by the SO_2 reduction of τ_M . This implies, in particular, that $HF^+(M, \mathfrak{t}) \cong \mathbb{Z}$ [OS4], Lemma 5.5.

The duality between $F_{X_+^\circ, \mathfrak{s}_+^\circ}^+: HF^+(M, \mathfrak{t}) \rightarrow HF^+(S^3)$ and $F_{X_+^\circ, \mathfrak{s}_+^\circ}^-: HF^-(S^3) \cong HF^-(-S^3) \rightarrow HF^-(-M, \mathfrak{t}) \cong HF^-(M, \mathfrak{t})$ (see Theorem 3.5 in [OS3]) implies that

$$[F_{X_+^\circ, \mathfrak{s}_+^\circ}^+(1 \otimes x_+)]_0 = F_{X_+^\circ, \mathfrak{s}_+^\circ}^{\text{mix}}(\theta_- \otimes x_+),$$

which gives

$$F_{X_+^\circ, \mathfrak{s}_+^\circ}^{\text{mix}}(\theta_- \otimes x_+) F_{X_-^\circ, \mathfrak{s}_-^\circ}^{\text{mix}}(\theta_- \otimes x_-) = \sum_{\mathfrak{s}|_{X_\pm^\circ} = \mathfrak{s}_\pm^\circ} \Phi_{X, \mathfrak{s}}(x_- \otimes x_+)$$

On the other hand, applying the product formula [OS3], to $X_{\pm}^{\odot} = X_{\pm}^{\odot} \cup N_{\pm}^{\odot}(\Sigma)$, viewed as a product cobordism of $X_{\pm}^{\odot}: S^3 \rightarrow M$ and $N_{\pm}^{\odot}: M \rightarrow S^3$, we obtain

$$F_{X_{\pm}^{\odot}, \mathfrak{s}_{\pm}^{\odot}}^{\text{mix}}(\theta_{-} \otimes x_{\pm}) = \sum_{\mathfrak{s}|_{X_{\pm}^{\odot}} = \mathfrak{s}^{\odot}} \Phi_{X_{\pm}, \mathfrak{s}_{\pm}}(x_{\pm})$$

using that the second cobordism induces an isomorphism from $HF^{+}(M, \mathfrak{t}) \cong \mathbb{Z}$ to $HF_0^{+}(S^3)$ (see [OS4], Theorem 5.3).

The structures $\mathfrak{s} \in \text{Spin}^{\mathbb{C}}(X_{\pm})$ in the latter sum differ just by multiples of the class $\sigma \in H^2(X_{\pm})$ Poincare-dual to $[\Sigma]$, and, thus, have distinct degrees, $d(\mathfrak{s})$, since $d(\mathfrak{s} + n\sigma) = d(\mathfrak{s}) + n\chi(\Sigma)$. In the other words, the latter formula is a decomposition of $F_{X_{\pm}^{\odot}, \mathfrak{s}_{\pm}^{\odot}}^{\text{mix}}$ into a sum of its homogeneous components (this idea was used in the proof of Lemma 5.6 in [OS4]). Passing from $\Phi_{X, \mathfrak{s}}$ to the dual $OS_X(\mathfrak{s})$ and comparing the components of the same degree, we obtain A4.

§4. PROOF OF THEOREMS 1.4.1

4.1. Independence of the choice of W . Consider a pair of Lefschetz fibrations, $W_i \rightarrow S^2$, $i = 1, 2$, with a fiber Σ such that $H_1(W_i) = 0$, and denote by $\mathfrak{r}_i \in \text{Spin}^{\mathbb{C}}(W_i, \Sigma)$ the canonical structures. Let $Y_i = X \#_{\Sigma} W_i$ and $W = W_1 \#_{\Sigma} W_2$, then $Z = X \#_{\Sigma} W \cong X \#_{\Sigma} W_1 \#_{\Sigma} W_2 \cong Y_1 \#_{\Sigma} W_2$.

4.1.1. Proposition. *For any $\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X, \Sigma)$ we have*

$$S_{Y_1}(\mathfrak{r} \#_{\Sigma} \mathfrak{r}_1) = S_Z(\mathfrak{r} \#_{\Sigma} \mathfrak{r}_1 \#_{\Sigma} \mathfrak{r}_2) = S_{Y_2}(\mathfrak{r} \#_{\Sigma} \mathfrak{r}_2)$$

Proof. Since the two equalities are analogous, we prove only the first one. Let $W_i^{\circ} = W_i \setminus N_i$, $i = 1, 2$, denote the complements of an open tubular neighborhood N_i of a fiber $\Sigma \subset W_i$, and $Y_i^{\circ} = X \#_{\Sigma} W_i^{\circ}$, $W^{\circ} = W_1^{\circ} \#_{\Sigma} W_2$. Applying Corollary 3.1.1 to a fiber sum $Y_1 \#_{\Sigma} W_2$ we obtain

$$S_{Y_1}(\mathfrak{r} \#_{\Sigma} \mathfrak{r}_1) = \sum_{h \in \Delta_M} S_Z((\mathfrak{r} \#_{\Sigma} \mathfrak{r}_1 \#_{\Sigma} \mathfrak{r}_2) + h).$$

where $M = \partial W_2^{\circ}$ and $\Delta_M \subset H^2(Z)$ is the image of $H^1(\Sigma)$ under the composition $\delta_M \circ q^*: H^1(\Sigma) \rightarrow H^1(M) \rightarrow H^2(Z)$, like in 2.11.

Note that the sum in the above formula has only one non-vanishing term, namely, with $h = 0$, because the restriction of any basic structure $(\mathfrak{r} \#_{\Sigma} \mathfrak{r}_1 \#_{\Sigma} \mathfrak{r}_2) + h$ to W° should coincide with $\mathfrak{r}_1 \#_{\Sigma} \mathfrak{r}_2|_{W^{\circ}}$ according to A3(3) and 2.12.1. On the other hand, for $h \neq 0$ it does not coincide, because of the following observation.

4.1.2. Lemma. *The following composition is injective*

$$H^1(\Sigma) \xrightarrow{q^*} H^1(M) \xrightarrow{\delta_M} H^2(Z) \longrightarrow H^2(W^{\circ})$$

(the last map here is the inclusion homomorphism).

Proof. In the Poincare dual homology homomorphisms

$$H_1(\Sigma) \rightarrow H_2(M) \rightarrow H_2(Z) \rightarrow H_2(W^{\circ}, \partial W^{\circ})$$

a class $h_1 \in H_1(\Sigma)$ is sent to the image of $h_2 = h_1 \times [S^1] \in H_2(\Sigma \times S^1) \cong H_2(M)$ under the inclusion map $H_2(M) \rightarrow H_2(W^\circ)$ composed with the relativization map $H_2(W^\circ) \rightarrow H_2(W^\circ, \partial W^\circ)$. The condition that $H_1(W_i) = H_1(W_i^\circ) = 0$ allows to find a cycle in $H_2(W^\circ)$ having non-vanishing intersection index with h_2 in W , if $h_1 \neq 0$, thus proving non-vanishing of the image of $[h_2]$ in $H_2(W^\circ, \partial W^\circ)$. \square

4.2. Proof of Properties (1)–(5) in Theorems 1.4.1.

(1) This property is just A1 applied to $X \#_\Sigma W$.

(2) This holds by definition of the invariants in the case of $\Sigma^2 > 0$.

(3) Note that the connected sum $X \#_\Sigma W$ is the same as the sum $X \#_{-\Sigma} \overline{W}$, where \overline{W} is the conjugate to W Lefschetz fibration. Axiom A2 implies that $S_{X,\Sigma}(\mathbf{r}) = S_{X \#_\Sigma W}(\mathbf{r} \#_\Sigma \mathbf{r}_{W,\Sigma})$ is equal to $\pm S_{X \#_\Sigma W}(\overline{\mathbf{r} \#_\Sigma \mathbf{r}_{W,\Sigma}})$ where the conjugate $\text{Spin}^\mathbb{C}$ structure $\overline{\mathbf{r} \#_\Sigma \mathbf{r}_{W,\Sigma}}$ equals to $\overline{\mathbf{r}} \#_{-\Sigma} \overline{\mathbf{r}_{W,\Sigma}}$ as follows from 2.12, and $\overline{\mathbf{r}_{W,\Sigma}} = \mathbf{r}_{\overline{W},\overline{\Sigma}}$, as remarked in 2.8. On the other hand, using \overline{W} to evaluate $S_{X,-\Sigma}(\overline{\mathbf{r}})$, we obtain $S_{X,-\Sigma}(\overline{\mathbf{r}}) = S_{X \#_{-\Sigma} \overline{W}}(\overline{\mathbf{r}} \#_{-\Sigma} \mathbf{r}_{\overline{W},\overline{\Sigma}})$ that is $\pm S_{X,\Sigma}(\mathbf{r})$.

(4) It follows immediately from Proposition 2.12.1 and A3(1), after we blowup X to obtain a Lefschetz fibration from a pencil.

(5) It follows from Corollary 3.1.1 applied to the fiber sum $X \#_\Sigma W$, namely

$$S_X(\mathfrak{s}) = \sum_{\mathbf{r} \in \text{abs}_{X,\Sigma}^{-1}(\mathfrak{s})} S_{X \#_\Sigma W}(\mathbf{r} \#_\Sigma \mathbf{r}_{W,\Sigma}) = \sum_{\mathbf{r} \in \text{abs}_{X,\Sigma}^{-1}(\mathfrak{s})} S_{X,\Sigma}(\mathbf{r}),$$

where $\mathbf{r}_{W,\Sigma} \in \text{Spin}^\mathbb{C}(W, \Sigma)$ is the canonical relative $\text{Spin}^\mathbb{C}$ structure of a Lefschetz fibration and $\text{abs}_{X,\Sigma}: \text{Spin}^\mathbb{C}(X, \Sigma) \rightarrow \text{Spin}^\mathbb{C}(X)$ the forgetful map. \square

4.3. Proof of the product formula (6). Consider a fiber sum $X = X_+ \#_\Sigma X_-$ and Lefschetz fibrations $W_\pm \rightarrow S^2$ with a fiber Σ and $H^1(W_\pm) = 0$. Put $Y_\pm = X_\pm \#_\Sigma W_\pm$, $W = W_+ \#_\Sigma W_-$, and $Z = X \#_\Sigma W \cong Y_+ \#_\Sigma Y_-$.

Choose a pair of structures $\mathbf{r}_\pm \in \text{Spin}^\mathbb{C}(X_\pm, \Sigma)$, denote by $\mathbf{r}_{W_\pm, \Sigma} \in \text{Spin}^\mathbb{C}(W_\pm, \Sigma)$ the canonical $\text{Spin}^\mathbb{C}$ structures of the Lefschetz fibrations and let $\mathfrak{s}_\pm = \mathbf{r}_\pm \#_\Sigma \mathbf{r}_{W_\pm, \Sigma} \in \text{Spin}^\mathbb{C}(Y_\pm)$, $\mathfrak{s} = \mathbf{r}_+ \#_\Sigma \mathbf{r}_{W_+, \Sigma} \#_\Sigma \mathbf{r}_- \#_\Sigma \mathbf{r}_{W_-, \Sigma} \in \text{Spin}^\mathbb{C}(Z)$. By Proposition 2.12.1, $\mathbf{r}_{W, \Sigma} = \mathbf{r}_{W_-, \Sigma} \vee \mathbf{r}_{W_+, \Sigma} \in \text{Spin}^\mathbb{C}(W, \Sigma)$ is the canonical relative $\text{Spin}^\mathbb{C}$ structure of the Lefschetz fibration in W and thus $\mathfrak{s} = (\mathbf{r}_+ \vee \mathbf{r}_-) \#_\Sigma \mathbf{r}_{W, \Sigma}$.

The product formula A4 applied to the fiber sum $Z = Y_+ \#_\Sigma Y_-$ reads

$$\sum_{k \in \mathbb{Z}} S_{Y_+}(\mathfrak{s}_+ + k\sigma_+) S_{Y_-}(\mathfrak{s}_- - k\sigma_-) = \sum_{\mathfrak{s}' \in \mathfrak{s}_+ \#_\Sigma \mathfrak{s}_-} S_Z(\mathfrak{s}')$$

The sum in the right-hand side contains only one term $S_Z(\mathfrak{s})$ which follows from the arguments in 4.1. On the other hand, $S_{X_\pm, \Sigma}(\mathbf{r}_\pm \pm k\sigma_\pm) = S_{Y_\pm}(\mathfrak{s}_\pm \pm k\sigma_\pm)$ and $S_{X, \Sigma}(\mathbf{r}_+ \vee \mathbf{r}_-) = S_Z(\mathfrak{s})$ by definition.

Equivalence of the two formulations of the product formula in 1.4.1(6) follows from that $\mathbf{r}'_+ \vee \mathbf{r}'_- = \mathbf{r}_+ \vee \mathbf{r}_-$ if and only if $\mathbf{r}'_\pm = \mathbf{r}_\pm \pm k\sigma_\pm$ for some $k \in \mathbb{Z}$. \square

4.4. Proof of the Adjunction inequality (7). The idea of the proof is to find an appropriate Lefschetz fibration $W \rightarrow S^2$ with a fiber $\Sigma_W \cong \Sigma$ having a membrane, $F_W \subset W$ whose boundary, ∂F_W matches with the boundary ∂F of membrane $F \subset X$.

Then after gluing F and F_W we can get a closed surface $\hat{F} \subset X \#_\Sigma W$, which will be oriented if the orientations of ∂F and ∂F_W do match.

More precisely, we should glue the complements X° and W° of tubular neighborhoods $N \subset X$ of Σ and $N_W \subset W$ of Σ_W so that $F \cap \partial N$ is glued to $F_W \cap \partial N_W$. It is not difficult to see that connectedness of $\Sigma \setminus \partial F$ guarantees that we can find such a gluing map $\partial N \rightarrow \partial N_W$.

Finally, we want to make use of the adjunction inequality A5 for \hat{F} . This requires $\hat{F}^2 = F^2 + F_W^2 \geq 0$, which holds if we can find F_W with a sufficiently big self-intersection index. If we choose F_W so that $\mathfrak{r}_{W,\Sigma}[F_W] = 0$ for the canonical structure $\mathfrak{r}_{W,\Sigma}$ of the Lefschetz fibration, then $c_1(\mathfrak{r} \# \mathfrak{r}_{W,\Sigma})[\hat{F}] = \mathfrak{r}[F] + \mathfrak{r}_{W,\Sigma}[F_W] = \mathfrak{r}[F]$ and thus

$$-\chi(\hat{F}) = -\chi(F) - \chi(F_W) \geq F^2 + F_W^2 + |\mathfrak{r}[F]|$$

which gives (7) of Theorem 1.4.1 provided $F_W^2 = -\chi(F_W)$. So, we reduced the problem to constructing the following example of W and F_W .

4.4.1. Proposition. *Let Σ be a surface of genus $g \geq 1$, $L \subset \Sigma$ an oriented curve (possibly multi-component) with the connected complement $\Sigma \setminus L$, and $n \in \mathbb{N}$. Then there exists a relatively minimal Lefschetz fibration $p: W \rightarrow \mathbb{CP}^1$, with $H_1(W) = 0$, whose fiber, Σ , has a membrane, $F_W \subset W$, such that*

- (1) $\partial F_W = L$ (as an oriented curve),
- (2) $F_W^2 = -\chi(F_W)$,
- (3) $\mathfrak{r}_{W,\Sigma}[F_W] = 0$, where $\mathfrak{r}_{W,\Sigma} \in \text{Spin}^{\mathbb{C}}(W, \Sigma)$ is the canonical $\text{Spin}^{\mathbb{C}}$ structure,
- (4) $-\chi(F_W) > n$.

4.5. Real Lefschetz fibrations. We will construct a complex algebraic Lefschetz fibration $p: W \rightarrow \mathbb{CP}^2$ endowed with a *real structure*, that is just an anti-holomorphic involution (the *complex conjugation in W*), $c: W \rightarrow W$, which commutes with p and the complex conjugation in \mathbb{CP}^1 . The *real locus*, \mathbb{RW} , of W is the fixed point set of c . For a real fiber, $\Sigma = p^{-1}(b)$, $b \in \mathbb{RP}^1$, we let $\mathbb{R}\Sigma = \Sigma \cap \mathbb{RW}$. A membrane F_W in our example will be the closure of a properly chosen connected component of $\mathbb{RW} \setminus \mathbb{R}\Sigma$ bounded by $\mathbb{R}\Sigma$. Such a choice guarantees the condition (2) of the Proposition 4.4.1, since the tangent bundle to \mathbb{RW} is anti-isomorphic to the normal bundle via the operator $J: \tau_X \rightarrow \tau_X$ of the complex structure. The condition (3) follows from that the real determinant gives a section trivializing the complex determinant bundle (and thus the associated $\text{Spin}^{\mathbb{C}}$ determinant), provided $F_W \subset \mathbb{RW}$ is orientable.

Note furthermore that the pairs (Σ, L) are classified up to homeomorphism respecting the orientations of Σ and L just by the genus g and the number of components, $r \leq g$, of L , under our assumption that $\Sigma \setminus L$ is connected. So, we will achieve $(\Sigma, \mathbb{R}\Sigma) \cong (\Sigma, L)$, that is the condition (1), if $\mathbb{R}\Sigma$ does not divide Σ into halves and has the required number r of the components. This reduces Proposition 4.4.1 to the following construction.

4.5.1. Proposition. *For any integers $k \in \mathbb{N}$ and $g \geq r \geq 1$, there exists a relatively minimal real algebraic Lefschetz fibration, $p: W \rightarrow \mathbb{CP}^1$, $\text{conj}_W: W \rightarrow W$, with $H_1(W) = 0$, and a real fiber $\Sigma = p^{-1}(b)$ of genus g such that*

- (1) $\mathbb{R}\Sigma$ has r components, and $\Sigma \setminus \mathbb{R}\Sigma$ is connected,
- (2) there is a connected orientable component of $\mathbb{RW} \setminus \mathbb{R}\Sigma$, whose closure, F_W , is bounded by $\mathbb{R}\Sigma$,
- (3) $g(F_W) \geq k$.

4.6. Proof of Proposition 4.5.1. Consider a double covering $q: W \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$, branched along a non-singular complex curve \mathbb{CA} defined over \mathbb{R} and having degree $(2g + 2, 2d)$, where d is sufficiently large. The Lefschetz fibration that we need is the composition of q with the projection to the first factor. A generic fiber, $\Sigma_t = q^{-1}(t \times \mathbb{CP}^1)$, $t \in \mathbb{CP}^1$, projects to \mathbb{CP}^1 as a double cover branched at $(2g + 2)$ points, $\mathbb{CA}_t = \mathbb{CA} \cap (t \times \mathbb{CP}^1)$, and thus has genus g . If this branching locus has $2r$ real points, $\mathbb{RA}_t = \mathbb{CA}_t \cap \mathbb{RP}^1$, then Σ_t satisfies the condition (1) of Proposition 4.5.1.

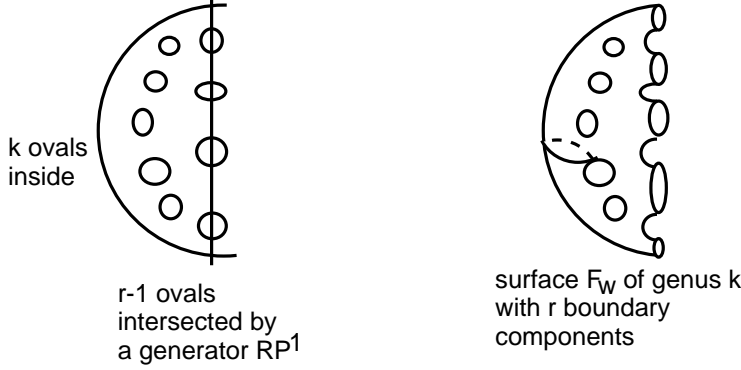


FIGURE 1

The conditions (2)–(3) will be obviously satisfied if we choose the curve \mathbb{RA} and a generator $t \times \mathbb{RP}^1$ with the mutual position as on the Figure 1. The curve with such position can be easily constructed by a perturbation of a nodal curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ splitting into a union of generators. \square

§5. SOME GENERALIZATIONS

5.1. Symplectic summand W instead of a Lefschetz fibration. It is an interesting question if the property A3(3) holds for arbitrary symplectic pairs (W, Σ) , $\Sigma^2 = 0$, which are relatively minimal (no exceptional curves in $X \setminus \Sigma$). If so, then the class of possible auxiliary summands in the definition of $S_{X, \Sigma}$ can be extended from Lefschetz fibrations to such relatively minimal symplectic pairs (W, Σ) , with $b_1(W) = 0$. Moreover, the second part of the normalization property of Theorem 1.4.1(4) would also be extended from Lefschetz fibrations to symplectic pairs.

5.2. Variants of $S_{X, \Sigma}$. One can consider a version of the invariant $S_{X, \Sigma}$,

$$S_{X, \Sigma, K}: \text{Spin}^{\mathbb{C}}(X, \Sigma)/K \rightarrow R_{X, \Sigma, K}$$

which depends on a subgroup $K \subset H^1(\Sigma)$. Here $\text{Spin}^{\mathbb{C}}(X, \Sigma)/K$ is the quotient of $\text{Spin}^{\mathbb{C}}(X, \Sigma)$ by the action of K , where $h \in K$ acts as $\mathfrak{s} \mapsto \mathfrak{s} + \delta(h)$, and $\delta: H^1(\Sigma) \rightarrow H^2(X, \Sigma)$ is the boundary homomorphism. The ring $R_{X, \Sigma, K}$ is $\mathbb{A}_{X, \Sigma, K}^* = \Lambda(H^1(X, \Sigma)/K) \otimes \mathbb{Z}[U]$ in the case of $S_X = SW_X$ and $\mathbb{A}_{X, \Sigma, K}^* \otimes \mathbb{Z}/2$ in the case of $S_X = OS_X$. Considering the invariant in the reduced form, we let $R_{X, \Sigma, K}$ be just \mathbb{Z} for SW and $\mathbb{Z}/2$ for OS invariants.

The definition of $S_{X, \Sigma, K}$ is similar to that of $S_{X, \Sigma}$, except that the condition $H_1(W) = 0$ for a Lefschetz fibration $W \rightarrow S^2$ is replaced by the condition $K = \text{Im}(H^1(W) \rightarrow H^1(\Sigma))$

(if K can be expressed as such an image). In particular, for $K = 0$, we have $S_{X,\Sigma,K} = S_{X,\Sigma}$ and for $K = H^1(\Sigma)$ the invariant $S_{X,\Sigma,K}$ coincides with the restriction, $S_{X,\mathfrak{s}_\Sigma}$, of the absolute invariant S_X .

In general, there is a splitting formula $S_{X,\Sigma,K}[\mathfrak{s}] = \sum_{\mathfrak{s}' \in [\mathfrak{s}]} S_{X,\Sigma}(\mathfrak{s}')$ or equivalently, $\underline{S}_{X,\Sigma,K} = (abs_K)_*(\underline{S}_{X,\Sigma})$, where $(abs_K)_*$ is the push-forward morphism of the projection $abs_K: \text{Spin}^{\mathbb{C}}(X, \Sigma) \rightarrow \text{Spin}^{\mathbb{C}}(X, \Sigma)/K$. The proof of this formula is analogous to the proof of (5) in Theorem 1.4.1.

5.3. A refinement of the Ozsváth-Szabó 4-dimensional invariant with respect to a mapping torus boundary component. The idea used in the definition of $S_{X,\Sigma}$ can be used also to define the refinement of the 4-dimensional Ozsváth-Szabó invariants in a more general setting. Assume for instance that $X: M_0 \rightarrow M_1$ is a cobordism between 3-manifolds, where $M_1 = M_f$ is a mapping torus of some homeomorphism $f: \Sigma \rightarrow \Sigma$.

The plane field tangent to the fibers of the projection $M_f \rightarrow S^1$ defines a canonical $\text{Spin}^{\mathbb{C}}$ extension of the tangent bundle τ_M . Let $\text{Spin}^{\mathbb{C}}(X, M_f)$ denote the set of the relative $\text{Spin}^{\mathbb{C}}$ structures in X with respect to such $\text{Spin}^{\mathbb{C}}$ extension over $M_f \subset \partial X$.

Choose any $\mathfrak{r} \in \text{Spin}^{\mathbb{C}}(X, M_1)$ and let $\mathfrak{s} = abs(\mathfrak{r}) \in \text{Spin}^{\mathbb{C}}(X)$, and $\mathfrak{t}_i = \mathfrak{s}|_{M_i}$, $i = 0, 1$ (here \mathfrak{t}_1 is the canonical structure on M_f). Consider an auxiliary cobordism $W: M_1 \rightarrow M_2$ which has structure of a Lefschetz fibration $q: W \rightarrow S^1 \times [1, 2]$ over the annulus, so that $M_i = q^{-1}(S^1 \times i)$. There is a canonical relative $\text{Spin}^{\mathbb{C}}$ structure, $\mathfrak{r}_W \in \text{Spin}^{\mathbb{C}}(W, M_1)$, which is a refinement of the canonical absolute structure $\mathfrak{s}_W = abs(\mathfrak{r}_W) \in \text{Spin}^{\mathbb{C}}(W)$. We assume that the Lefschetz fibration is relatively minimal and $b_1(W) = 0$ (one can always find such a fibration bounded by any prescribed mapping tori M_i , $i = 1, 2$; for example we may assume in addition that $M_2 \cong \Sigma \times S^1$).

The homomorphism

$$F_{X,\mathfrak{r}}^+: HF^+(M_0, \mathfrak{t}_0) \rightarrow HF^+(M_1, \mathfrak{t}_1) \cong \mathbb{Z}$$

is defined as the composition of $F_{X \cup W, \mathfrak{r} \mathfrak{r}_W}^+: HF^+(M_0, \mathfrak{t}_0) \rightarrow HF^+(M_2, \mathfrak{t}_2)$ and the inverse $(F_{W, \mathfrak{s}_W}^+)^{-1}$ to the isomorphism $F_{W, \mathfrak{s}_W}^+: HF^+(M_1, \mathfrak{t}_1) \rightarrow HF^+(M_2, \mathfrak{t}_2)$.

The action of $H_1(X \cup W) = H_1(X \cup W, W) = H_1(X, M_1)$ in $F_{X \cup W, \mathfrak{r} \mathfrak{r}_W}^+$ composed with the isomorphism $(F_{W, \mathfrak{s}_W}^+)^{-1}$ defines an action of \mathbb{A}_{X, M_1} in $F_{X, \mathfrak{r}}$.

The product formula of [OS3] applied to the cobordism $X \cup W: M_0 \rightarrow M_2$ implies a splitting $F_{X, \mathfrak{s}}^+ = \sum_{\mathfrak{r} \in abs^{-1}(\mathfrak{s})} F_{X, \mathfrak{r}}^+$. Similarly one can define maps $F_{X, \mathfrak{r}}^-$ and $F_{X, \mathfrak{r}}^{\text{mix}}$ and obtain analogous decompositions of $F_{X, \mathfrak{s}}^-$ and $F_{X, \mathfrak{s}}^{\text{mix}}$.

All the constructions in this section admit also similar versions for SW invariants.

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