EXOTIC EMBEDDINGS OF $\#6\mathbb{R}\mathrm{P}^2$ IN THE 4-SPHERE

S. Finashin

Abstract. We construct an infinite sequence of smooth embeddings of $\#6\mathbb{R}\mathrm{P}^2$ in S^4, which are all ambient homeomorphic, but pairwise ambient non-diffeomorphic. The double covers of S^4 ramified along these surfaces form a family of the exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}^2}$ constructed recently by Park, Stipsicz and Szabó.

§1. Introduction

Theorem A. *For any $k \geqslant 6$ there exists an infinite family of smoothly embedded surfaces $F_i \subset S^4$, $i = 1, 2, \dots$, homeomorphic to $F = \#k\mathbb{R}\mathrm{P}^2$ (connected sum of k copies of $\mathbb{R}\mathrm{P}^2$) and with the normal Euler number $F^2 = 2k-4$, such that*

(1) *the pairs (S^4, F_i) are all homeomorphic; the ambient homeomorphisms can be assumed to be diffeomorphisms in some neighborhoods of F_i;*
(2) *(S^4, F_i) are all pairwise non-diffeomorphic.*

Theorem A improves the result of [FKV1]-[FKV2], where a similar family of $F_i \subset S^4$ was constructed for $k = 9$. Our construction of F_i is based on similar ideas and makes use of the examples of exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ in [PSS]. More precisely, our goal can be stated as follows.

Theorem B. *There exists an infinite family of smoothly embedded surfaces $F_i \subset S^4$, $i = 1, 2, \dots$ which are all homeomorphic to a connected sum $\#6\mathbb{R}\mathrm{P}^2$, such that $\pi_1(S^4 \smallsetminus F_i) = \mathbb{Z}/2$, and the double covers $X_i \to S^4$ branched along F_i provide an infinite family of exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ constructed in* [PSS].

Recall the well-known diffeomorphism $\mathbb{C}\mathrm{P}^2/\operatorname{conj} = S^4$. The image of $\mathbb{R}\mathrm{P}^2$ in the quotient-space S^4 represent an isotopy class of *standard embeddings* of $\mathbb{R}\mathrm{P}^2$ with the normal Euler number -2. Another isotopy class of standard embeddings (with the normal Euler class 2) is obtained by reversing the orientation of S^4. It is represented by the image of $\mathbb{R}\mathrm{P}^2$ in $S^4 = \overline{\mathbb{C}\mathrm{P}}^2/\operatorname{conj}$, which will be denoted $\overline{\mathbb{R}\mathrm{P}}^2 \subset S^4$. A non-orientable surface $F \subset S^4$ will be called *standard* if it splits into an ambient connected sum of such standard embeddings, that is $F = p\mathbb{R}\mathrm{P}^2\#q\overline{\mathbb{R}\mathrm{P}}^2$.

Theorem B implies Theorem A. It is proven in [PSS] that X_i are pairwise non-diffeomorphic. This implies that the pairs (S^4, F_i) are non-diffeomorphic to each other. It is known moreover that for any $k \geqslant 1$ $X_i\#k\overline{\mathbb{C}\mathrm{P}}^2$, $i = 1, 2\dots$, remain pairwise non-diffeomorphic, which implies that $F_i\#k\overline{\mathbb{R}\mathrm{P}}^2 \subset S^4$ are also all ambient non-diffeomorphic.

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The values $F_i^2 = 8$ of the normal Euler numbers for F_i in Theorem B can be obtained from the the signature formula $\sigma(X_i) = -4 = 2\sigma(S^4) - \frac{1}{2}F_i^2$. For the connected sum $F_i \# k\overline{\mathbb{RP}}^2$ the Euler number becomes $8 + 2k$.

It follows from [FKV2] that the obstruction for ambient homeomorphism of F_i belongs to a finite group. This implies that we can choose infinitely many ambient homeomorphic surfaces F_i in the infinite set of pairwise non-diffeomorphic ones, which are covered by exotic $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$. $\square$

Remark. In [K], it is shown that the finite ambiguity observed in [FKV2] for the exotic $10\#\mathbb{RP}^2$ actually vanishes. This means that all the examples of embedded $10\#\mathbb{RP}^2$ that were constructed in [FKV2] are actually homeomorphic to a standard $\mathbb{RP}^2 \# 9\overline{\mathbb{RP}}^2$. It seems probable that the arguments in [K] can be adopted after an appropriate modification in our case of $\#6\mathbb{RP}^2$. This would imply that all the examples of F_i in Theorem B are actually ambient homeomorphic to a standard $\mathbb{RP}^2 \# 5\overline{\mathbb{RP}}^2$.

Scheme of the proof of theorem B. There are several alternative constructions of an exotic $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$ in [PSS], and the one suitable for us is obtained by some surgery from a rational elliptic surface, X, with a fiber of type $\mathbb{I}_8$. The first step is a double node neighborhood knot surgery on X, which yields a 4-manifold X_K containing a nodal *pseudo-section.* Next, X_K is blown up at several points so that we obtain a suitable chain of spheres, $C = C_1 \cup \dots C_k$, which can be rationally blowdown on the last step. Our aim is to perform these constructions equivariantly.

In §2, we construct a special example of a rational elliptic surface, X, with a fiber $\mathbb{I}_8$, which is defined over reals, and thus has an involution of the complex conjugation, c. It is essential for the further constructions that the components of the $\mathbb{I}_8$-fiber as well as the four remaining singular fishtail fibers are all *real* (that is invariant under the complex conjugation). In §4, we observe that a certain non-singular real fiber, T, which is constructed in §2, can be used for an equivariant double node knot surgery. We check that the nodal pseudo-section S_K obtained after such a surgery can be chosen invariant under the involution. Following the construction in [PSS], we blowup several points, which turn out to be all real in our example of X. Finally we obtain as in [PSS] a chain of spheres $C = C_1 \cup \dots \cup C_n$, whose components are all c-invariant.

In §3, we discuss an equivariant blowdown of such chains C. It is crucial for us that the quotient X/c turns out to be S^4 and that the quotient by the involution remains the same as we modify the 4-manifold and the involution. So, all the involved equivariant transformations of X (knot surgery, blowup at a real point and rational blowdown) just modify the fixed point set F in the quotient S^4. Another crucial fact is that the fundamental group $\pi_1(S^4 \smallsetminus F) = \mathbb{Z}/2$ is preserved under these modifications of F.

More precisely, π_1 is preserved after a rational blowdown of C if we put a certain condition on C. This condition is satisfied for two of the configurations proposed in Proposition 2.5 of [PSS]: for $C_{79,44}$ and for $C_{89,9}$, which are the chains
$(-2, -5, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -3)$ and
$(-10, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -2, -2, -2, -2, -2, -2)$.

Following the construction [PSS] in the equivariant setting, we obtain a certain 4-manifold $\widehat{X}$ homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$, with an involution, $\widehat{c}\colon \widehat{X} \to \widehat{X}$. In the quotient $\widehat{X}/\widehat{c} = S^4$ there is a surface $F \subset S^4$ which is the fixed point set of $\widehat{c}$. Observing that F is connected, non-orientable (because $F^2 = 8 \neq 0$), and estimating its Euler characteristic $\chi(F) = 2\chi(S^4) - \chi(\widehat{X})$, we deduce that it is $\#6\mathbb{R}\mathrm{P}^2$.

A sequence of the twist-knots K_i that can be used for the knot surgery on the first step of the construction (see figure 6a) yields a sequence $\widehat{X}_i$ of exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ with involutions $\widehat{c}_i$, and a sequence of surfaces $F_i \subset S^4$ required for theorem B. $\square$

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§2. Real rational elliptic surfaces with special singular fibers

2.1. Double planes ramified along quartics. Recall that the double covering over $\mathbb{C}\mathrm{P}^2$ branched along a non-singular quartic, $A \subset \mathbb{C}\mathrm{P}^2$, yields a del Pezzo surface $X_A = \mathbb{C}\mathrm{P}^2\#7\overline{\mathbb{C}\mathrm{P}}^2$. A pencil of lines, $L_s \subset \mathbb{C}\mathrm{P}^2$, centered at $x \in \mathbb{C}\mathrm{P}^2$, is covered by an elliptic pencil, $T_s \subset \mathbb{C}\mathrm{P}^2\#7\overline{\mathbb{C}\mathrm{P}}^2$, whose singular fibers correspond to the lines tangent to A. Blowing up the pull-back of x in $\mathbb{C}\mathrm{P}^2\#7\overline{\mathbb{C}\mathrm{P}}^2$, we obtain an elliptic fibration, $p\colon X \to \mathbb{C}\mathrm{P}^1$.

Assume that the quartic A has a singular point, $y \in A$, of the type $\mathbb{A}_n$ (by definition at such a singularity p has a local model $(z_1, z_2) \mapsto z_1^{n+1} + z_2^2$), and the basepoint x is generic. Then we obtain an elliptic fibration with a singular fiber T_0 of the type $\mathbb{I}_{n+1}$ (in Kodaira's classification). The fiber T_0 corresponds to the line $L_0 \subset \mathbb{C}\mathrm{P}^2$ passing through x and y. If in addition L_0 is tangent to A at the basepoint $x \in A$, then the fiber T_0 is of the type $\mathbb{I}_{n+3}$.

For instance, we obtain $\mathbb{I}_8$-fiber T_0 if we choose a quartic $A = A_1 \cup A_3$ that splits into a cubic, A_3, and a real line, A_1, tangent to A_3 at its inflection point, y (see Figure 1a). The corresponding line L_0 should pass through y and be tangent to A_3 at some other point, x, which will be the center of the pencil of the lines L_s.

2.2. Construction of a special real elliptic fibration. Consider the double covering $q\colon X_A \to \mathbb{C}\mathrm{P}^2$ ramified along a degree $2n$ curve $A \subset \mathbb{C}\mathrm{P}^2$ having an equation $f = 0$ with real coefficients. The complex conjugation in $\mathbb{C}\mathrm{P}^2$ can be lifted to X_A in two ways. Namely, there are two real algebraic models of X_A defined by a weighted homogeneous equation $y^2 = \pm f(x_0, x_1, x_2)$ in a quasi-projective space $P_{1,1,1,n}$ with the coordinates x_0, x_1, x_2, y of weights $1, 1, 1, n$. The corresponding involutions $c_\pm\colon X_A \to X_A$, induced from the complex conjugation in $P_{1,1,1,n}$, have fixed point sets $\mathrm{Fix}(c_\pm|_{X_A})$ which are projected by q to the regions $\mathbb{R}\mathrm{P}^2_{\pm f} = \{x \in \mathbb{R}\mathrm{P}^2 \mid \pm f(x) \geqslant 0\}$ bounded by the curve $A_\mathbb{R} = A \cap \mathbb{R}\mathrm{P}^2$.

In our example of the real quartic $A = A_1 \cup A_3$, we choose the region $\mathbb{R}\mathrm{P}^2_f$ as is shown on Figure 1c) and the corresponding involution $c_A\colon X_A \to X_A$ whose fixed point set is $q^{-1}(\mathbb{R}\mathrm{P}^2_f)$.

Blowing up the singularity and the two infinitely near base-points of X_A, we obtain a *real elliptic fibration*, $p\colon X \to \mathbb{C}\mathrm{P}^1$, endowed with an involution $c\colon X \to X$ commuting with p. Let $F = \mathrm{Fix}(c)$ denote its fixed point set.

Lemma 1. *The real elliptic fibration $p\colon X \to \mathbb{C}\mathrm{P}^1$ constructed above has the following properties.*

(1) *X contains a real singular fiber $T_0 = C_1 \cup \dots \cup C_8$ of the type $\mathbb{I}_8$, whose components C_i are c-invariant.*

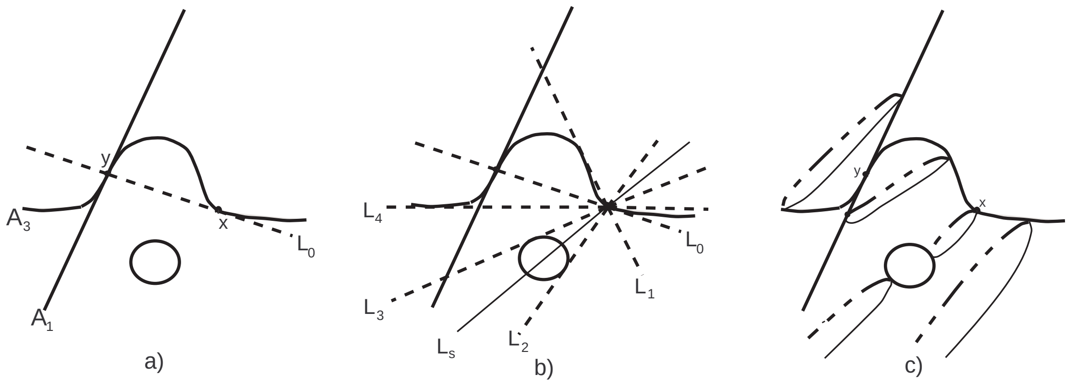

FIGURE 1
a) Quartic $A = A_1 \cup A_3$ and the basepoint x. b) The tangent lines L_i, $i = 1, 2, 3, 4$ of the pencil and line L_s corresponding to the fiber $T = T_s$. c) The real locus of the double plane X_A.

(2) *X contains 4 other c-invariant singular fibers, T_i, $i = 1, 2, 3, 4$, which are ordinary fishtails.*
(3) *The elliptic fibration $p\colon X \to \mathbb{C}\mathrm{P}^1$ admits a c-invariant section.*
(4) *X can be blowdown to $\mathbb{C}\mathrm{P}^2$, so that each of the nine consecutively contracted (-1)-curves is real. This transforms the non-singular real fibers F_s to non-singular real cubics.*
(5) *$X/c = S^4$.*
(6) *If the singular fibers T_i, $i = 0, 1, 2, 3, 4$ are the pre-images of the tangent lines L_i on Figure 1b, then a non-singular real fiber $T = T_s$ chosen between T_2 and T_3 has real locus, $T \cap F$, formed by two connected components, like is shown on Figure 2c. The two vanishing curves in T, which are contracted as T is degenerated into the singular fibers T_2 and T_3, are isotopic. A vanishing curve from this isotopy class can be chosen c-invariant, and so that c reverses its orientation.*
(7) *The complement $F \smallsetminus (F \cap T)$ is connected.*

Proof. The real form of the singularity $\mathbb{A}_5$ involved in our construction has local model $x^6 - y^2 + z^2 = 0$, and c-invariance of the components C_i is verified by its analysis. The fibers T_i in are c-invariant because the tangent lines L_i are real.

To justify (3), we will present 6 real sections. The first one is the exceptional curve, $E_x \subset X$, which appears after the second of the two infinitely near blowups at the basepoint $x \in X_A$, as we construct X. Another section is the proper transform, $A_1^* \subset X$, of the line A_1. The proper transform of a real line $L' \subset \mathbb{C}\mathrm{P}^2$ passing through y and tangent to A_3 (see Figure 2a) splits in two components, L_1', L_2', which are both sections of p. Another tangent line L'' shown on Figure 2a give similarly components L_1'' and L_2''.

Let us choose the cyclic order of the components C_i of T_0, and suppose that C_1 intersects E_x. Then A_1^* intersects C_5, and the curves L_1' and L_1'' intersect C_7 and C_3. We may suppose that L_1' intersect C_7, like it is shown on Figure 2. If we blowdown consecutively E_x, C_1, C_2, C_3, C_4, A_1^*, L_1', C_7, and C_6, then we obtain $\mathbb{C}\mathrm{P}^2$, which proves (4). (Note that the remaining components, C_5 and C_8, will represent a line and a conic in $\mathbb{C}\mathrm{P}^2$ obtained after 9 blowdowns.)

We can deduce (5) from (4) using that $\mathbb{C}\mathrm{P}^2/\operatorname{conj} = S^4$, which implies that a blowup at a real point effects to the quotient as a connected sum with $S^4 = \overline{\mathbb{C}\mathrm{P}}^2/\operatorname{conj}$, and thus, topologically does not change the quotient.

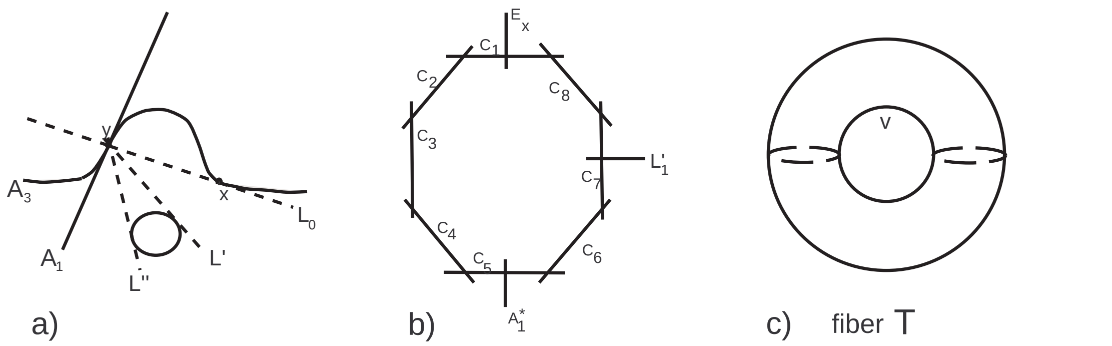

FIGURE 2

a) Tangent lines L' and L''. b) Fiber T_0 and 3 disjoint real sections E_x, A_1^*, and L_1'. c) The fixed point set of c dividing the fiber T, and the vanishing curve v.

Inspecting Figure 1c we observe that the fixed point set $T \cap F$ of the complex conjugation acting on T has two connected components, as it is shown on Figure 2c. This can be understood from Figure 1c. The type of the vanishing curves on T is determined by the types of the real critical points of the projection $F \to \mathbb{R}\mathrm{P}^1$ (restriction of p). The critical points in the fibers T_2 and T_3 have both index one, which implies that the homology class of the corresponding vanishing curves belong to the (-1)-eigenspace of c_* in $H_1(T)$. This determines these vanishing curves up to isotopy, thus, proving (6).

Property (7) is clear from Figure 1c), if we take into account connectedness of the real locus of the corresponding singularity $x^6 - y^2 + z^2 = 0$ after its resolution. □

§3. Equivariant Rational Blowdown

3.1. Rational blow-down surface surgery. Let X denote a smooth 4-manifold with *a chain of spheres* $C = C_1 \cup \dots \cup C_k \subset X$, which intersect each other consecutively and transversely, so that their dual weighted graph is a chain-tree sketched on Figure 3. A regular neighborhood, $N(C)$, of C is a plumbing 4-manifold, P_C, corresponding to this weighted graph.

Certain chains C can be *rationally blowdown*, that is we can remove $N(C) = P_C$ from X and replace it by some rational homology ball, Q_C, with the same boundary $\partial Q_C = \partial N(C)$. This gives a new 4-manifold $\widehat{X} = X' \cup Q_C$, where $X' = X \smallsetminus N(C)$.

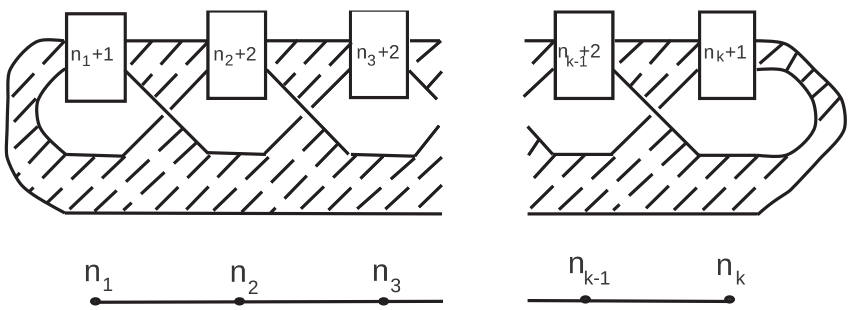

FIGURE 3

The plumbing surface F_C described by a chain-tree can be presented as the span-surface of a two-bridge knot diagram. The numbers in the boxes count the half-twists.

It is well known and easy to see that P_C can be described as the double cover over D^4 branched along a surface, F_C, obtained by plumbing of the unknotted bands,

$F_{n_i} \subset D^4$, $i = 1, \dots, k$, where n_i stands for the framing of the band (number of its half-twists which is taken with sign "$-$" in the case of left-hand half-twists). Such a plumbing surface can be sketched as is shown on Figure 3.

As it is observed in [FS1], Q_C is the double cover of D^4 branched along another surface, $R_C \subset D^4$, bounded by the same link as F_C, $L_C = \partial R_C = \partial F_C$. More details about surface R_C can be extracted from [CH], and we will only mention that R_C is obtained by connecting a pair of disjoint unknotted discs, $D_1 \sqcup D_2$, via a ribbon in S^3 and then pushing in inside D^4 the interior of the surface. This ribbon connects either ∂D_1 with ∂D_2, or the boundary of one of the discs to itself, say of ∂D^1. In the first case we obtain a knotted disc in D^4. In the second case we obtain a disjoint union of a disc D_2 with a Möbius band, as it is shown on Figure 4b in the simplest example. (One can show that the band cannot be orientable, because ∂F_C cannot have 3 boundary components.)

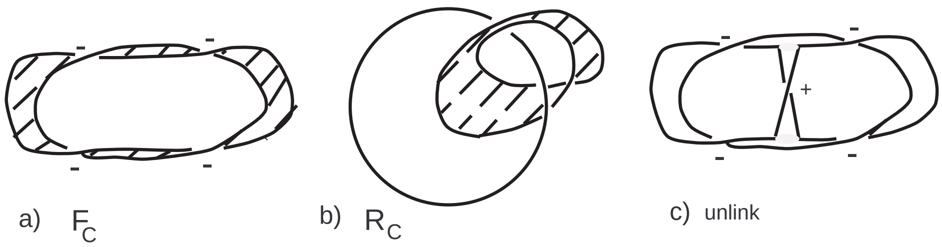

FIGURE 4

a) A band $F_C = F_{-4}$ with the four negative half-twists (marked with the signs "$-$"). b) R_C (a disc and a band) bounded by the same link L_C. c) L_C differs from an unlink by a ribbon move.

Lemma 2. *Consider a rational blowdown of a chain of spheres $C \subset X$, which yields $\widehat{X} = X' \cup R_C$. Assume that X is endowed with an orientation preserving involution $c\colon X \to X$, which keeps each of the components, $C_i \subset C$ invariant, and reverses its orientation, so that* $\mathrm{Fix}(c) \cap C_i \neq \varnothing$*. Then, if $N(C)$ is chosen c-invariant, the rational blowdown can be made equivariantly, which yields an involution $\widehat{c}\colon \widehat{X} \to \widehat{X}$.*

Such a blowdown gives the same quotient $Y = X/c = \widehat{X}/\widehat{c}$. The fixed point sets $F = \mathrm{Fix}\, c$ *and* $\widehat{F} = \mathrm{Fix}(\widehat{c})$ *descended to Y give the same locus $F' = F \cap Y' = \widehat{F} \cap Y'$, inside $Y' = X'/c$. Furthermore, $N(C)/c = D^4$, and $F \cap D^4$ is isotopic to the plumbing surface F_C. The piece of surface $\widehat{F} \cap D^4$ is isotopic to the surface R_C.*

Proof. Under these assumptions, $c|_{N(C)}$ is equivalent to the deck transformation of the double branched covering $N(C) \to D^4$. The involution $\widehat{c}$ just extends the involution $c|_{X \smallsetminus N(C)}$ to Q_C as the deck transformation of the branched covering $Q_C \to D^4$. $\square$

We say that $(\widehat{X}, \widehat{c})$ is obtained by an *equivariant rational blowdown* from (X, c).

3.2. The characteristic sub-configuration. It is not difficult to see that the number of components of the link $L_C = \partial F_C$ is determined by the parity of numbers C_i^2, namely, L_C has one component if the intersection matrix $(C_i \circ C_j)$ is non-singular modulo 2, and has two component if singular.

We say that the union of some of the components C_i forms a *characteristic sub-configuration*, $W \subset C$, if the fundamental class $[W] \in H_2(C; \mathbb{Z}/2)$ is a Wu element of the intersection form $(C_i \circ C_j)$, that is $C_i^2 = C_i \circ W \mod 2$ for all $i \in \{1, \dots, k\}$. It is easy to see that the characteristic sub-configuration W is unique if the matrix $(C_i \circ C_j)$ is non-singular modulo 2 (has odd determinant). If this matrix is $\mod 2$ singular, then there are two characteristic sub-configurations, W and W', whose sum gives the non-trivial element of the null-space of $(C_i \circ C_j)$ in $H_2(C; \mathbb{Z}/2)$ (it is easy to check that this null-space has dimension at most 1).

Remark. It is simple to determine W using an orientation of the link diagram of L_C. Namely, W contains C_i if and only if the opposite sides of the band F_{n_i} are co-directed, like is shown on Figure 5. This follows from that $W \cap F$ realizes the first Stiefel-Whitney class $w_1(F_C)$.

3.3. Commutativity of π_1 after rational blowdowns. Suppose that a sphere $C_0 \subset X$ extends the chain C to a longer chain $\widetilde{C} = C_0 \cup C_1 \cup \dots \cup C_n$. This means that C_0 is a c-invariant sphere (like the other C_i) which intersects C_1 transversely at a single point and does not intersect C_i, if $i > 1$.

Lemma 3. *Assume that*

(1) *the link $L(C)$ is a knot,*
(2) *a characteristic sub-configuration $W \subset C$ is not characteristic for $\widetilde{C}$.*

Then $\pi_1(Y \smallsetminus \widehat{F}) = \pi_1(Y \smallsetminus F)$.

Remark. The second assumption of the lemma means that C_1 is not included into W if $n_0 = C_0^2$ is odd, and $C_1 \subset W$ if n_0 is even.

Proof. Applying the Van Kampen theorem, we see that $\pi_1(Y \smallsetminus F) = G' *_{G_L} G_C$, where $G' = \pi_1(Y' \smallsetminus F')$, $G_C = \pi_1(D^4 \smallsetminus F_C)$, and $G_L = \pi_1(S^3 \smallsetminus L_C)$. Similarly, $\pi_1(Y \smallsetminus \widehat{F}) = G' *_{G_L} \widehat{G}$, where $\widehat{G} = \pi_1(D^4 \smallsetminus R_C)$.

The plan of the proof is to observe that the inclusion homomorphisms $G_L \to G_C$, $G_L \to \widehat{G}$ are epimorphisms, and that their kernels, K and $\widehat{K}$, vanish under the inclusion homomorphism $G_L \to G'$. This implies that the homomorphisms $G' \to G' *_{G_L} G_C$ and $G' \to G' *_{G_L} \widehat{G}$ are isomorphisms.

First of all, note that the upper Wirtinger presentation for the link L_C diagram shown on Figure 3 implies that its group G_L is generated by two elements a, b, presented by the loops around the overpasses ℓ_a and ℓ_b shown of Figure 5.

The homomorphism $G_L \to G_C$ is epimorphic because G_C is a cyclic group (since F_C is a connected span-surface for L_C, whose interior is pushed out from S^3 inside D^4). Inspecting the homology, we see that $G_C = \mathbb{Z}$ is obtained from G_L by adding the relation $a = b$, in the case of orientable surface F_C. If F_C is non-orientable, then $G_C = \mathbb{Z}/2$ is obtained from G_L by adding two relations: $a = b$ and $a = b^{-1}$. These relations generate K.

The homomorphism $G_L \to \widehat{G}$ is also epimorphic, because R_C is a ribbon-surface. The kernel $\widehat{K}$ is contained in the commutator subgroup $[G_L, G_L]$, which is the kernel of the product map $G_L \to \widehat{G} \to H_1(D^4 \smallsetminus R_C) = H_1(S^3 \smallsetminus L_C) = \mathbb{Z}$, where the latter two equalities are due to our assumption that L_C is connected, and so R_C is a disc. Thus, $[G_L, G_L]$ is generated by the relation $a = b$ if overpasses ℓ_a and ℓ_b on Figure 5 inherit co-directed orientation from L_C, and by the relation $a = b^{-1}$ if these overpasses inherit opposite orientations.

Showing that the images of a, b under the inclusion homomorphisms $G_L \to G'$ (for which we keep the same notation a, b) satisfy the both relations $a = b$ and $a = b^{-1}$, will complete the proof.

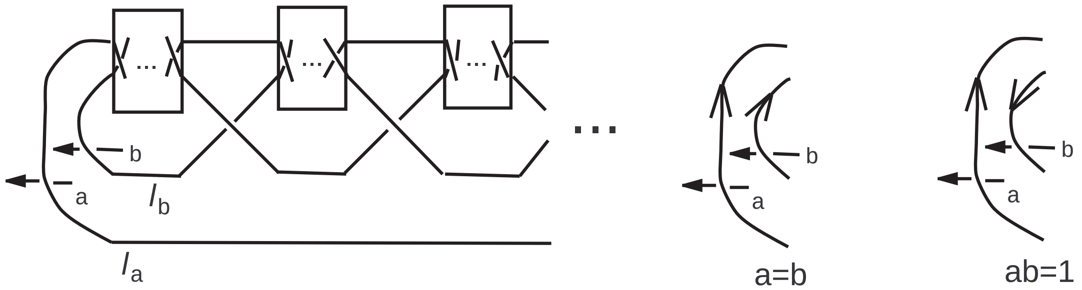

Figure 5

Overpasses ℓ_a, ℓ_b and the corresponding generators a and b of $\pi_1(S^3 \smallsetminus L_C)$. The case of co-directed and oppositely directed overpasses ℓ_a, ℓ_b, with the corresponding relations between a and b (after adding the commutativity relation $ab = ba$)

One of these relations comes from a regular neighborhood $N(D_0)$ of the disc $D_0 = C_0/c$ in Y. Note that $H = N(D_0) \cap Y'$ is a 4-ball containing an unknotted disc $F_H = F \cap H$, so that $\pi_1(H \smallsetminus F_H) = \mathbb{Z}$. The common piece of boundary of H and D^4 is a 3-ball, which intersects F along a pair of arcs, $\ell_a \cup \ell_b$. It is not difficult to see that in $\pi_1(H \smallsetminus F_H)$ we obtain the relation $a = b$ if n_0 is even, and $a = b^{-1}$ if odd. With this relation, group G_L becomes abelian, and we obtain another relation (which depends on the orientation of ℓ_a, ℓ_b induced from L_C, as was explained). Under the second assumption of our lemma, these two relations are different, that is the both relations $a = b$ and $a = b^{-1}$ are satisfied in $\pi_1(Y \smallsetminus \widehat{F})$. $\square$

§4. Equivariant version of the Fintushel-Stern double node knot surgery

4.1. Equivariant knot surgery. The 4-dimensional knot surgery consists in removing from a 4-manifold X a trivialized tubular neighborhood $N(T) = T \times D^2$ of a torus $T \subset X$, and replacing it by $S^1 \times C(K)$, where $C(K)$ is a knot complement (see [FS1]). It is supposed that the gluing map $S^1 \times \partial C(K) \to T \times \partial D^2$ identifies a longitude $\mathrm{pt} \times \ell \subset S^1 \times C(K)$ with a meridian of T, $m_T = \mathrm{pt} \times \partial D^2 \subset \partial N(T)$, which yields a 4-manifold X_K, homologically equivalent to X.

In the equivariant version of this construction, we suppose that X is endowed with an orientation preserving involution, c, which keeps invariant the torus T as well as its neighborhood $N(T)$. We say that a trivialization $N(T) = T \times D^2$ of a tubular neighborhood, $N(T)$, of T is *equivariant* if the action of c on $N(T)$ is presented as the direct product of $c|_T$ and the complex conjugation in $D^2 \subset \mathbb{C}$. Note that equivariant trivializability is equivalent to existence of a projection $N(T) \to D^2$ which commutes with $c|_{N(T)}$ and the complex conjugation in D^2. In the case of our interest, T is a real non-singular fiber in a real elliptic fibration and thus admits such an equivariantly trivializable neighborhood.

Let us assume in addition that $c|_T$ reverses orientation of T and has two-component fixed point set, $F \cap T$ (see Figure 2c). In this case the quotient $\mathcal{A} = T/c$ is an annulus and in the coordinates defined by some diffeomorphism $T = S^1 \times S^1$ the action of c looks as $(z_0, z_1) \mapsto (z_0, \bar{z}_1)$. Thus for an appropriate diffeomorphism $N(T) = S^1 \times S^1 \times D^2$, this action looks as $(z_0, z_1, z_2) \mapsto (z_0, \bar{z}_1, \bar{z}_2)$.

From a knot $K \subset S^3$ we require that it has an axis of symmetry, and intersects this axis at a pair of points. It will be convenient to choose the complex conjugation, conj, in $S^3 \subset \mathbb{C}^2$ as such a symmetry, so that the axis is $S^1_{\mathbb{R}} = S^3 \cap \mathbb{R}^2$. Such a knot K admits an equivariant tubular neighborhood, $N(K)$ in which c acts as

$(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$, with respect to a trivialization $S^1 \times D^2 = N(K)$. We can choose such a trivialization to be *null-framed*, which means that a longitude $\ell_K = S^1 \times \mathrm{pt}$ is null-homologous in the knot complement $C(K) = \mathrm{Cl}(S^3 \smallsetminus N(K))$.

We can glue $S^1 \times C(K)$ to $X \smallsetminus N(T)$ via an equivariant gluing map $g\colon S^1 \times \partial C(K) \to \partial N(T)$. Using the coordinates (z_0, z_1, z_2) in $\partial C(K) = \partial N(K)$ and $\partial N(T)$ from the above trivializations of $N(K)$ and $N(T)$, we define map g as $(z_0, z_1, z_2) \mapsto (z_0, z_2, z_1)$. Such an equivariant knot surgery yields a 4-manifold X_K endowed with an involution, $c_K\colon X_K \to X_K$.

4.2. The tangle surgery in the quotient-spaces. The quotient $N(K)/\operatorname{conj}$ is a 3-ball, which can be viewed as a regular neighborhood, $N(\mathfrak{s}_K)$, of the arc $\mathfrak{s}_K = K/\operatorname{conj}$ in $S^3 = S^3/\operatorname{conj}$. Thus, $B_K = C(K)/\operatorname{conj}$ is also a 3-ball, complementary to $N(\mathfrak{s}_K)$. The unknot $S^1_{\mathbb{R}} \subset S^3/\operatorname{conj}$ splits into a trivial tangle $\mathfrak{t} = S^1_{\mathbb{R}} \cap N(s_K)$ and a non-trivial one, $\mathfrak{t}_K = S^1_{\mathbb{R}} \cap B_K$ (see Figure 6c).

Example 1. The twist-knot $K = K_n$, which will be used in our construction, admits a conj-invariant presentation, as it is sketched on Figure 6a. Figures 6d–6e present the corresponding tangle splitting $S^1_{\mathbb{R}} = \mathfrak{t} \cup \mathfrak{t}_K \subset S^3$.

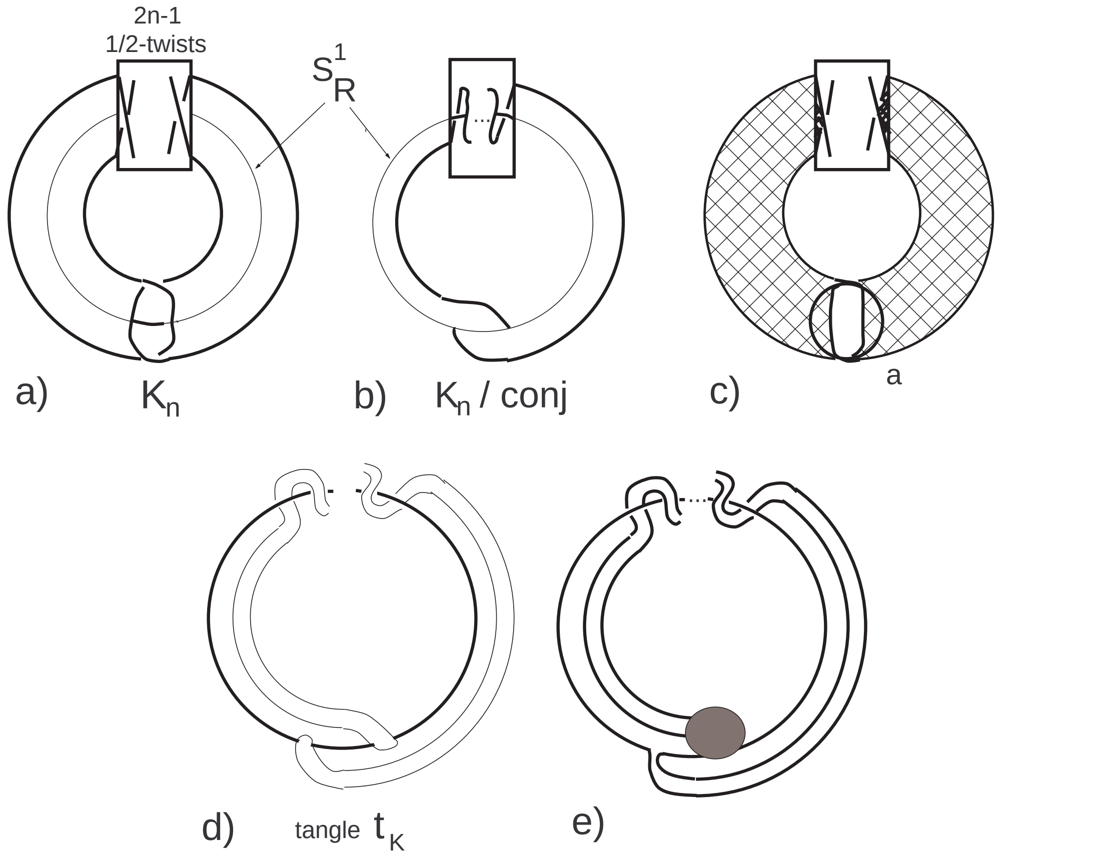

FIGURE 6

a) The twist-knot $K = K_n$ with the axis of symmetry $S^1_{\mathbb{R}}$. b) The arc $\mathfrak{s}_K = K/\operatorname{conj}$. c) Seifert surface S°_K of genus 1 bounded by K. It contains conj-invariant curve a, which bounds a conj-invariant disc D_a. d) The ball $N(\mathfrak{s}_K)$ and tangle $\mathfrak{t}_K$ in its complement. e) $\mathfrak{t}_K$ after deformation of $N(\mathfrak{s}_K)$ (the ball shaded on the figure).

The quotient-space X_K/c_K is obtained from X/c by removing a regular neighborhood, $N = N(\mathcal{A})$, of the annulus $\mathcal{A} = T/c$, which can be viewed as $N = S^1 \times N(\mathfrak{s}_K) = S^1 \times B^3$, and replacing it by $S^1 \times B_K = S^1 \times B^3$. Such a surgery does not change the topological type of a 4-manifold, so we can identify the both quotients, $Y = X/c = X_K/c_K$.

The branching locus F_K of the double covering $X_K \to Y$ is obtained from F after replacing $F_N = F \cap N = S^1 \times \mathfrak{t}$ by $S^1 \times \mathfrak{t}_K$ inside $N = S^1 \times B^3$. Note that the components of $\mathfrak{t}_K$ and of $\mathfrak{t}$ connect the same pairs of their common endpoints. We denote these four endpoints $p_1^{\pm}, p_2^{\pm}$, and assume that p_i^+ is connected with p_i^-. Moreover, the both tangles must have *the same framing.* This means by definition that the kernel of the inclusion homomorphism $H_1(S^2 \smallsetminus \partial\mathfrak{t}) \to H_1(D^3 \smallsetminus \mathfrak{t})$ is the same as for $H_1(S^2 \smallsetminus \partial\mathfrak{t}_K) \to H_1(D^3 \smallsetminus \mathfrak{t}_K)$.

Such kind of surgery will be called *tangle surgery of $F \subset Y$ along an annulus membrane $\mathcal{A}$. It can be applied to any surface F in a 4-manifold Y under the assumption that the annulus membrane $\mathcal{A}$ with the boundary on this surface is null-framed. This means by definition that for some trivialization $N = S^1 \times D^3$ of its regular neighborhood, $N = N(\mathcal{A})$, the part of surface $F \cap N$ is identified with $S^1 \times \mathfrak{t}$, and $\mathcal{A}$ is identified with $S^1 \times \mathfrak{s}$, where $\mathfrak{s}$ is a line segment connecting the midpoints of the components of* $\mathfrak{t}$ *(see Figure 7a). The following lemma summarizes our observations.*

Lemma 4. *An equivariant knot surgery on (X, c) along a c-invariant torus $T \subset X$ gives (X_K, c_K) with the same quotient-space $Y = X_K/c_K = X/c$. The fixed point set $F_K \subset Y$ of c_K is obtained from the fixed point set F by the tangle surgery along the annulus membrane $\mathcal{A} = T/c$.* □

4.3. Commutativity of π_1 throughout the knot surgery.

Lemma 5. *Assume that $F \subset Y$ is a surface in a 4-manifold and $\mathcal{A}$ is a null-framed annulus membrane on F such that $F \smallsetminus \partial\mathcal{A}$ is connected. Assume that F_K is obtained from F by applying the tangle surgery along $\mathcal{A}$ with respect to the $\mathfrak{t}_K$, where $K = K_n$ is the twist-knot from Example 1. Assume furthermore that the group $\pi_1(Y \smallsetminus (F \cup \mathcal{A}))$ is abelian. Then $\pi_1(Y \smallsetminus F_K)$ is also abelian and isomorphic to $\pi_1(Y \smallsetminus F)$.*

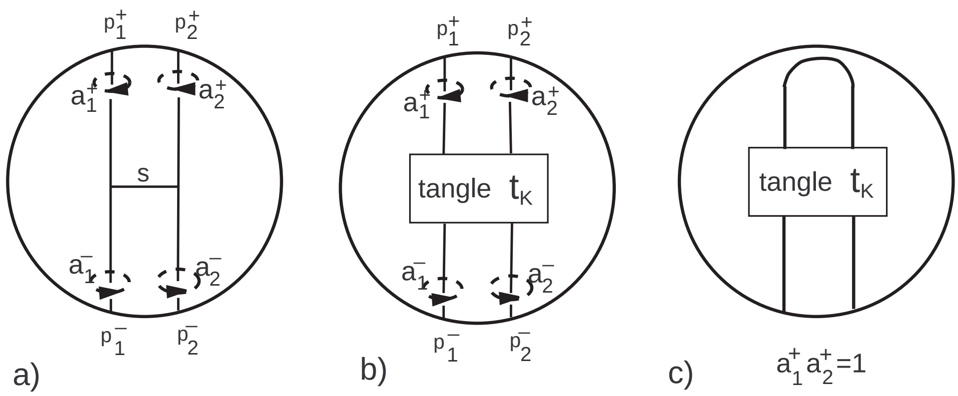

FIGURE 7

a) Trivial tangle $\mathfrak{t}$ with the connecting line segment $\mathfrak{s}$. The generators $a_i^{\pm}$ of $\pi_1(S^2 \smallsetminus \partial\mathfrak{t})$. b) The result of a tangle surgery. c) Adding the relation $a_1^+ a_2^+ = 1$ to the group of tangle $\mathfrak{t}_K$ effects like connecting together the points p_1^+ and p_2^+.

Proof. Let $Y' = \mathrm{Cl}(Y \smallsetminus N)$ and $F' = F \cap Y'$, $F_{N,K} = F_K \cap N$. By the Van Kampen theorem, $\pi_1(Y \smallsetminus F_K) = G' *_H G_N$, where $G' = \pi_1(Y' \smallsetminus F')$, $H = \pi_1(\partial N \smallsetminus \partial F_N)$, and $G_N = \pi_1(N \smallsetminus F_{N,K})$. Note that $N \smallsetminus (\mathcal{A} \cup F_N)$ can be deformation retracted to its boundary $\partial N \smallsetminus \partial F_N$, which implies that group G' is abelian, by the assumption on $\pi_1(Y \smallsetminus (F \cup \mathcal{A})$. Note that $G' *_H G_N = G' *_H (G_N/K)$, where K is the image in G_N of the kernel of the homomorphism $H \to G'$. We will show that G_N/K is an abelian group and the product homomorphism $H \to G_N \to G_N/K$ is epimorphic. This implies that $G' *_H G_N$ is a quotient of G' and thus is also abelian.

Note that $H = \mathbb{Z} \times \pi_1(S^2 \smallsetminus \partial\mathfrak{t})$, where the second factor is a free group or rank 3. It is convenient to present this free group by 4 generators, $a_1^\pm$, $a_2^\pm$, satisfying the relation $a_1^+ a_1^- a_2^- a_2^+ = 1$. These generators correspond to the loops around the tangle endpoints, $p_1^\pm, p_2^\pm \in S^2$, in the positive direction, see Figure 7a.

Let us fix some element $a \in G'$ presented by a loop around $F \smallsetminus \partial\mathcal{A}$. Commutativity of G' and connectedness of $F \smallsetminus \partial\mathcal{A}$ imply that the inclusion homomorphism $H \to G'$ sends each of $a_i^\pm$ either to a, or to a^{-1} (depending on the topology of the boundary $\partial\mathcal{A}$ as an oriented curve in F). Such a relation, $a_i^\pm = a$ or $a_i^\pm = a^{-1}$, is inherited by the quotient-group G_N/K. To complete proof of the lemma it is enough to show that by adding this relation we transform group $\pi_1(D^3 \smallsetminus \mathfrak{t}_K)$ into a cyclic group with a generator a (since the factor $\mathbb{Z}$ in $G_N = \mathbb{Z} \times \pi_1(D^3 \smallsetminus \mathfrak{t}_K)$ lies in the center and comes from the corresponding factor in $H = \mathbb{Z} \times \pi_1(S^2 \smallsetminus \{p_1^+, p_1^-, p_2^+, p_2^-\}))$.

We will present two arguments. The first one can be applied to any knot K admissible for an equivariant knot surgery, but it works only if we have a relation $a_1^+ a_2^+ = 1$, or $a_1^- a_2^- = 1$. Note that if we connect the endpoints p_1^+ and p_2^+ as is shown on Figure 7c, we modify the group $\pi_1(B^3 \smallsetminus \mathfrak{t}_K)$ by adding a relation $a_1^+ a_2^+ = 1$. In the case of tangles $\mathfrak{t}_K$ constructed from conj-invariant knots K, this modification transforms the tangle into an unknotted arc in D^3. Thus, the group $\pi_1(D^3 \smallsetminus \mathfrak{t}_K)$ becomes cyclic and generated by any of the elements $a_i^\mp$. The case of adding relation $a_1^- a_2^- = 1$ is analogous.

Our second argument is specific for the twist-knot $K = K_n$, but can be applied in the case of relation $a_1^+ = a_2^+$ or $a_1^- = a_2^-$ (as well as in the case of relation $a_1^\pm a_2^\pm = 1$ considered before). First, we observe that the upper Wirtinger presentation gives 5 generators for $\pi_1(D^3 \smallsetminus \mathfrak{t}_K)$, namely $a_i^\pm$, $i = 1, 2$, and one more generator b shown on Figure 8.

The two strands of the tangle $\mathfrak{t}_K$ with the origins at the points p_1^+ and p_2^+ pass together several times under $S^1_\mathbb{R}$. These underpasses separate the consecutive overpasses on the first strand which yield elements $a_1^+, b^{-1}a_1^+b, \dots$, which are all conjugate to a_1^+. The similar overpasses on the second strand give elements $a_2^+, b^{-1}a_2^+b, \dots$, which are conjugate to a_2^+ by the same sequence of elements. In the end of the sequence, we obtain elements $b^{-1} = x^{-1}a_1^+x$ and $(a_1^-)^{-1} = x^{-1}a_2^+x$, which are conjugate to a_1 and a_2 via the same element $x \in \pi_1(D^3 \smallsetminus \mathfrak{t}_K)$. So, a relation $a_1^+ = a_2^+$ implies that $b = a_1^-$, whereas $a_1^+ a_2^+ = 1$ implies $ba_1^- = 1$. In any case, generator b can be eliminated, and after adding two more relations to $\pi_1(D^3 \smallsetminus \mathfrak{t}_K)$, namely $a_1^- = a_1^+$ (or $a_1^- = (a_1^+)^{-1}$) and $a_2^- = a_1^+$ (or $a_2^- = (a_1^+)^{-1}$, we obtain a cyclic group.

Finally, we can observe that surface F_K is homologically equivalent to F, and thus $H_1(Y \smallsetminus F_K) = H_1(Y \smallsetminus F)$. Since the group $\pi_1(Y \smallsetminus F)$ is obviously abelian due to the assumption of the lemma, we obtain an isomorphism $\pi_1(Y \smallsetminus F_K) = \pi_1(Y \smallsetminus F)$. $\square$

Lemma 6. *Let (X, c) be the real elliptic surface constructed in §2, and T the real fiber from Lemma 1(6). Then membrane $\mathcal{A} = T/c$ satisfies the assumptions of Lemma 5, and thus $\pi_1(S^4 \smallsetminus F_K) = \mathbb{Z}/2$ (here $S^4 = X/c$ by Lemma 1(6)).*

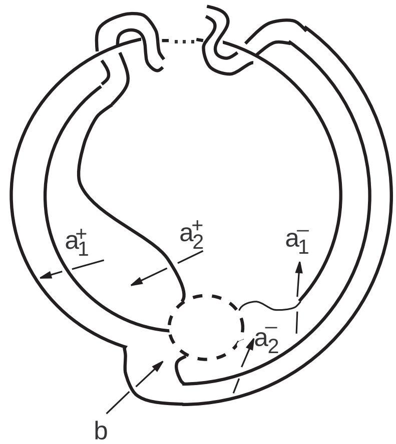

FIGURE 8

The tangle group becomes abelian after adding the relations $a_1^+ = a_2^+$, $a_1^- = (a_1^+)^{\pm 1}$, and $a_2^- = (a_1^+)^{\pm 1}$.

Proof. Connectedness of $F \smallsetminus \mathcal{A}$ is observed in Lemma 1(4).

The group $\pi_1(S^4 \smallsetminus (F \cup \mathcal{A}))$ was shown to be cyclic in [FKV2], §4, under the assumption that $T \subset X$ is obtained from a real non-singular cubic in $\mathbb{C}\mathrm{P}^2$ by blowing up the base-points of a real pencil of cubics. This is so in our case, as follows from property (6) of Lemma 1. $\square$

4.4. The equivariant double node surgery. To justify that a pseudo-section $S_K \subset X_K$ can be chosen c-invariant we recall first its construction in [FS2]. Consider a disc $\Delta_1 \subset \mathbb{C}\mathrm{P}^1$ which contains inside precisely two critical values $s_+, s_- \subset \Delta_1$ of an elliptic Lefschetz fibration $p\colon X \to \mathbb{C}\mathrm{P}^1$. Assume moreover that the corresponding two vanishing curves in a non-singular fiber, T_s, $s \in \Delta_1$, are isotopic. Let $\Delta \subset \Delta_1$ denote a smaller disc not containing points $s_\pm$, and $U = p^{-1}(\Delta)$. Consider a section $S \subset X$ of p. Its restriction over Δ is the disc $\widetilde{\Delta} = S \cap U$. Since the gluing map in the definition of the knot surgery which yields X_K may be changed by an isotopy, we can make the boundary $\partial\widetilde{\Delta}$ match with the boundary of a Seifert surface $S_K^\circ \subset C(K) \times \mathrm{pt} \subset X_K$ and obtain a closed surface $S_K^* = (S \smallsetminus \widetilde{\Delta}) \cup S_K^\circ$ in X_K. If $K = K_n$ (the twist-knot on Figure 6a), then S_K^* is a torus which has a certain disc membrane $D_a^* \subset X_K$ bounded by curve $a = \partial D_a^* = D_a^* \cap S_K^*$ and having self-intersection $(D_a^*)^2 = -1$ (relative to the boundary on the surface S_K^*). The torus S_K^* can be deformed into a fishtail $S_K \subset X_K$, as we pinch curve $a \subset S_K^*$ along disc D_a^*. The local topology of S_K near its singular point is like near an algebraic double point, and the embedded surface $S_K \subset X_K$ is topologically equivalent to a rational curve with a single node and self-intersection $(S_K)^2 = -1$.

To construct disc D_a^*, we first take a disc $D_a \subset S^3$ bounded by a, so that D_a intersects K at a pair of points (see Figure 6c). Disc D_a punctured at these points is embedded in $pt \times C_K \subset S^1 \times C_K \subset X_K$. The boundary of the punctures are curves in two different fibers of p, and by definition of our knot surgery these curves are vanishing (corresponding to the singular values $s_\pm$) and so can be filled by the discs centered at the nodes of the singular fibers over $s_\pm$.

Lemma 7. *Consider the real elliptic surface (X, c) constructed in §2. Let $K = K_n$ be the twist-knot embedded* conj*-invariantly in S^3, as is shown on Figure 6a. Assume that (X_K, c_K) is obtained from (X, c) by an equivariant knot surgery along* *the non-singular fiber $T = T_s$ specified in Lemma 1(6). Then, the pseudo-section S_K can be chosen c_K-invariant.*

Proof. By Lemma 1(3), we can suppose that section S is c-invariant. We consider a disc $\Delta \ni s$ which is invariant under the complex conjugation in $\mathbb{C}\mathrm{P}^1$, and denote by $r_\pm$ the endpoints of the interval $\Delta \cap \mathbb{R}\mathrm{P}^1$. In our example of real elliptic surface in §2, we have a pair of real critical values, $s_\pm$, whose fibers $T_2 = p^{-1}(s_-)$, $T_3 = p^{-1}(s_+)$ can be used for a double node knot surgery, as it follows from Lemma 1(6). The curve $S \cap \partial U$ is a conj-invariant longitude of the knot K in the boundary of $C_K \cong \mathrm{pt} \times C_K$. This longitude spans a conj-invariant Seifert surface $S_K^\circ \subset C_K$, as it is shown on Figure 6c. This implies that torus S_K^* can be chosen c_K-invariant. Furthermore, we can choose the disc D_a^* to be c_K-invariant as well. Namely, we choose first a conj-invariant disc D_a (see Figure 6c) and make conj-invariant punctures around the two intersection points $D_a \cap K$, which are both real. The boundary of such punctures are c-invariant curves in the two fibers $p^{-1}(r_\pm) \subset \partial U$, namely, the vanishing curves of the critical values $s_\pm$. If the discs filling these curves are chosen c-invariant, then the disc D_a^* becomes c-invariant as well.

Finally, note that there is a c_K-equivariant deformation of S_K, which contracts disc D_a^* and degenerates the torus S_K into S_K^*. Its construction goes like in the non-equivariant case: we deform s_K using a flow of a vector field tangent to D_a^*. To obtain an equivariant deformation, this field should be chosen c-invariant. $\square$

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Middle East Technical University, Department of Mathematics
Ankara 06531 Turkey
E-mail address: serge@metu.edu.tr